CHAPTER 5

COHOMOLOGY VIA FORMS

5.1 The DeRham cohomology groups of a manifold

In the last four chapters we've frequently encountered the question: When is a closed k-form on an open subset of $\mathbb{R}^N$ (or, more generally on a submanifold of $\mathbb{R}^N$) exact? To investigate this question more systematically than we've done heretofore, let X be an n-dimensional manifold and let

$$Z^k(X) = \{\omega \in \Omega^k(X)\,;\, d\omega = 0\} \tag{5.1.1}$$

and

$$B^k(X) = \{\omega \in \Omega^k(X)\,;\, \omega \text{ in } d\Omega^{k-1}(X)\} \tag{5.1.2}$$

be the vector spaces of closed and exact k-forms. Since (1.1.2) is a vector subspace of (1.1.1) we can form the quotient space

$$H^k(X) = Z^k(X)/B^k(X)\,, \tag{5.1.3}$$

and the dimension of this space is a measure of the extent to which closed forms fail to be exact. We will call this space the k^{th} *DeRham cohomology group of the manifold*, X. Since the vector spaces (1.1.1) and (1.1.2) are both infinite dimensional there is no guarantee that this quotient space is finite dimensional, however, we'll show later in this chapter that it is in lots of interesting cases.

The spaces (1.1.3) also have compactly supported counterparts. Namely let

$$Z^k_c(X) = \{\omega \in \Omega^k_c(X)\,;\, d\omega = 0\} \tag{5.1.4}$$

and

$$B^k_c(X) = \{\omega \in \Omega^k_c(X)\,,\, \omega \text{ in } d\Omega^{k-1}_c(X)\}\,. \tag{5.1.5}$$

Then as above $B^k_c(X)$ is a vector subspace of $Z^k_c(X)$ and the vector space quotient

$$H^k_c(X) = Z^k_c(X)/B^k_c(X) \tag{5.1.6}$$

is the k^{th} *compactly supported DeRham cohomology group of* X.

Given a closed k-form, $\omega \in Z^k(X)$, we will denote by $[\omega]$ the image of ω in the quotient space (1.1.3) and call $[\omega]$ the *cohomology class* of ω. We will also use the same notation for compactly supported cohomology. If ω is in $Z_c^k(X)$ we'll denote by $[\omega]$ the cohomology class of ω in the quotient space (1.1.6).

Some cohomology groups of manifolds we've already computed in the previous chapters (although we didn't explicitly describe these computations as "computing cohomology"). We'll make a list below of some of the things we've already learned about DeRham cohomology:

1. If X is connected, $H^0(X) = \mathbb{R}$. Proof: A closed zero form is a function, $f \in \mathcal{C}^\infty(X)$ having the property, $df = 0$, and if X is connected the only such functions are constants.

2. If X is connected and non-compact $H_c^0(X) = \{0\}$. Proof: If f is in $\mathcal{C}_0^\infty(X)$ and X is non-compact, f has to be zero at some point, and hence if $df = 0$ it has to be identically zero.

3. If X is n-dimensional,

$$\Omega^k(X) = \Omega_c^k(X) = \{0\}$$

for k less than zero or k greater than n, hence

$$H^k(X) = H_c^k(X) = \{0\}$$

for k less than zero or k greater than n.

4. If X is an oriented, connected n-dimensional manifold, the integration operation is a linear map

$$\int : \Omega_c^n(X) \to \mathbb{R} \tag{5.1.7}$$

and, by Theorem 4.8.1, the kernel of this map is $B_c^n(X)$. Moreover, in degree n, $Z_c^n(X) = \Omega_c^n(X)$ and hence by (1.1.6), we get from (1.1.7) a bijective map

$$I_X : H_c^n(X) \to \mathbb{R}\,. \tag{5.1.8}$$

In other words

$$H_c^n(X) = \mathbb{R}\,. \tag{5.1.9}$$

5. Let U be a star-shaped open subset of $\mathbb{R}^n$. In §2.5, exercises 4–7, we sketched a proof of the assertion: For $k > 0$ every closed form, $\omega \in Z^k(U)$ is exact, i.e., translating this assertion into cohomology language, we showed that

$$H^k(U) = \{0\} \text{ for } k > 0\,. \tag{5.1.10}$$

6. Let $U \subseteq \mathbb{R}^n$ be an open rectangle. In §3.2, exercises 4–7, we sketched a proof of the assertion: If $\omega \in \Omega_c^k(U)$ is closed and k is less than n, then $\omega = d\mu$ for some $(k-1)$-form, $\mu \in \Omega_c^{k-1}(U)$. Hence we showed

$$H_c^k(U) = 0 \text{ for } k < n\,. \tag{5.1.11}$$

7. *Poincaré's lemma for manifolds*: Let X be an n-dimensional manifold and $\omega \in Z^k(X)$, $k > 0$ a closed k-form. Then for every point, $p \in X$, there exists a neighborhood, U of p and a $(k-1)$-form $\mu \in \Omega^{k-1}(U)$ such that $\omega = d\mu$ on U. Proof: For open subsets of $\mathbb{R}^n$ we proved this result in §2.3 and since X is locally diffeomorphic at p to an open subset of $\mathbb{R}^n$ this result is true for manifolds as well.

8. Let X be the unit sphere, S^n, in $\mathbb{R}^{n+1}$. Since S^n is compact, connected and oriented

$$H^0(S^n) = H^n(S^n) = \mathbb{R}\,. \tag{5.1.12}$$

We will show that for $k \neq, 0, n$

$$H^k(S^n) = \{0\}\,. \tag{5.1.13}$$

To see this let $\omega \in \Omega^k(S^n)$ be a closed k-form and let $p = (0, \dots, 0, 1) \in S^n$ be the "north pole" of S^n. By the Poincaré lemma there exists a neighborhood, U, of p in S^n and a $k-1$-form, $\mu \in \Omega^{k-1}(U)$ with $\omega = d\mu$ on U. Let $\rho \in \mathcal{C}_0^\infty(U)$ be a "bump function" which is equal to one on a neighborhood, U_0 of U in p. Then

$$\omega_1 = \omega - d\rho\mu \tag{5.1.14}$$

is a closed k-form with compact support in $S^n - \{p\}$. However stereographic projection gives one a diffeomorphism

$$\varphi : \mathbb{R}^n \to S^n - \{p\}$$

(see exercise 1 below), and hence $\varphi^*\omega_1$ is a closed compactly supported k-form on $\mathbb{R}^n$ with support in a large rectangle. Thus by (1.1.14) $\varphi^*\omega = d\nu$, for some $\nu \in \Omega_c^{k-1}(\mathbb{R}^n)$, and by (1.1.14)

$$\omega = d(\rho\mu + (\varphi^{-1})^*\nu) \tag{5.1.15}$$

with $(\varphi^{-1})^*\nu \in \Omega_c^{k-1}(S^n - \{p\}) \subseteq \Omega^k(S^n)$, so we've proved that for $0 < k < n$ every closed k-form on S^n is exact.

We will next discuss some "pull-back" operations in DeRham theory. Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. For $\omega \in \Omega^k(Y)$, $df^*\omega = f^*\, d\omega$, so if ω is closed, $f^*\omega$ is as well. Moreover, if $\omega = d\mu$, $f^*\omega = df^*\mu$, so if ω is exact, $f^*\omega$ is as well. Thus we have linear maps

$$f^* : Z^k(Y) \to Z^k(X) \tag{5.1.16}$$

and

$$f^* : B^k(Y) \to B^k(X) \tag{5.1.17}$$

and comparing (1.1.16) with the projection

$$\pi : Z^k(X) \to Z^k(X)/B^k(X)$$

we get a linear map

$$Z^k(Y) \to H^k(X)\,. \tag{5.1.18}$$

In view of (1.1.17), $B^k(Y)$ is in the kernel of this map, so by Theorem 1.2.2 one gets an induced linear map

$$f^\sharp : H^k(Y) \to H^k(Y)\,, \tag{5.1.19}$$

such that $f^\sharp \circ \pi$ is the map (1.1.18). In other words, if ω is a closed k-form on Y $f^\sharp$ has the defining property

$$f^\sharp[\omega] = [f^*\omega]\,. \tag{5.1.20}$$

This "pull-back"operation on cohomology satisfies the following chain rule: Let Z be a manifold and $g : Y \to Z$ a $\mathcal{C}^\infty$ map. Then if ω is a closed k-form on Z

$$(g \circ f)^*\omega = f^*g^*\omega$$

by the chain rule for pull-backs of forms, and hence by (1.1.20)

$$(g \circ f)^\sharp[\omega] = f^\sharp(g^\sharp[\omega])\,. \tag{5.1.21}$$

The discussion above carries over verbatim to the setting of compactly supported DeRham cohomology: If $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map it induces a pull-back map on cohomology

$$f^\sharp : H_c^k(Y) \to H_c^k(X) \tag{5.1.22}$$

and if $f : X \to Y$ and $g : Y \to Z$ are proper $\mathcal{C}^\infty$ maps then the chain rule (1.1.21) holds for compactly supported DeRham cohomology as well as for ordinary DeRham cohomology. Notice also that if $f : X \to Y$ is a diffeomorphism, we can take Z to be X itself and g to be f^{-1}, and in this case the chain rule tells us that the maps (1.1.19) and (1.1.22) are bijections, i.e., $H^k(X)$ and $H^k(Y)$ and $H_c^k(X)$ and $H_c^k(Y)$ are isomorphic as vector spaces.

We will next establish an important fact about the pull-back operation, $f^\sharp$; we'll show that it's a *homotopy invariant* of f. Recall that two $\mathcal{C}^\infty$ maps

$$f_i : X \to Y\,, \quad i = 0, 1 \tag{5.1.23}$$

are homotopic if there exists a $\mathcal{C}^\infty$ map

$$F : X \times [0,1] \to Y$$

with the property $F(p,0) = f_0(p)$ and $F(p,1) = f_1(p)$ for all $p \in X$. We will prove:

Theorem 5.1.1. *If the maps (1.1.23) are homotopic then, for the maps they induce on cohomology*

$$f_0^\sharp = f_1^\sharp\,. \tag{5.1.24}$$

Our proof of this will consist of proving this for an important special class of homotopies, and then by "pull-back" tricks deducing this result for homotopies in general. Let v be a complete vector field on X and let

$$f_t : X \to X\,, \quad -\infty < t < \infty$$

be the one-parameter group of diffeomorphisms it generates. Then

$$F : X \times [0,1] \to X\,, \qquad F(p,t) = f_t(p)\,,$$

is a homotopy between f_0 and f_1, and we'll show that for this homotopic pair (1.1.24) is true. Recall that for $\omega \in \Omega^k(X)$

$$\left(\frac{d}{dt} f_t^* \omega\right)(t=0) = L_v = \iota(v)\, d\omega + \, d\iota(v)\omega$$

and more generally for all t

$$\begin{aligned}
\frac{d}{dt} f_t^* \omega &= \left(\frac{d}{ds} f_{s+t}^* \omega\right)(s=0) \\
&= \left(\frac{d}{ds} (f_s \circ f_t)^* \omega\right)(s=0) \\
&= \left(\frac{d}{ds} f_t^* f_s^* \omega\right)(s=0) = f_t^* \left(\frac{d}{ds} f_s^* \omega\right)(s=0) \\
&= f_t^* L_v \omega \\
&= f_t^* \iota(v)\, d\omega + \, d f_t^* \iota(v)\omega\,.
\end{aligned}$$

Thus if we set

$$Q_t \omega = f_t^* \iota(v) \omega \tag{5.1.25}$$

we get from this computation:

$$\frac{d}{dt} f^* \omega = \, dQ_t + Q_t \, d\omega \tag{5.1.26}$$

and integrating over $0 \le t \le 1$:

$$f_1^* \omega - f_0^* \omega = \, dQ\omega + Q\, d\omega \tag{5.1.27}$$

where

$$Q : \Omega^k(Y) \to \Omega^{k-1}(X)$$

is the operator

$$Q\omega = \int_0^1 Q_t \omega \, dt\,. \tag{5.1.28}$$

The identity (1.1.24) is an easy consequence of this "chain homotopy" identity. If ω is in $Z^k(X)$, $d\omega = 0$, so

$$f_1^* \omega - f_0^* \omega = \, dQ\omega$$

and

$$f_1^\sharp[\omega] - f_0^\sharp[\omega] = [f_1^*\omega - f_0^*\omega] = 0\,.$$

Q.E.D.

We'll now describe how to extract from this result a proof of Theorem 1.1.1 for *any* pair of homotopic maps. We'll begin with the following useful observation.

Proposition 5.1.2. *If $f_i : X \to Y$, $i = 0, 1$, are homotopic $\mathcal{C}^\infty$ mappings there exists a $\mathcal{C}^\infty$ map*

$$F : X \times \mathbb{R} \to Y$$

such that the restriction of F to $X \times [0,1]$ is a homotopy between f_0 and f_1.

Proof. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$, $\rho \geq 0$, be a bump function which is supported on the interval, $\frac{1}{4} \leq t \leq \frac{3}{4}$ and is positive at $t = \frac{1}{2}$. Then

$$\chi(t) = \int_{-\infty}^{t} \rho(s)\,ds \Bigg/ \int_{-\infty}^{\infty} \rho(s)\,ds$$

is a function which is zero on the interval $t \leq \frac{1}{4}$, is one on the interval $t \geq \frac{3}{4}$, and, for all t, lies between 0 and 1. Now let

$$G : X \times [0,1] \to Y$$

be a homotopy between f_0 and f_1 and let $F : X \times \mathbb{R} \to Y$ be the map

$$F(x,t) = G(x, \chi(t))\,. \tag{5.1.29}$$

This is a $\mathcal{C}^\infty$ map and since

$$F(x,1) = G(x,\chi(1)) = G(x,1) = f_1(x)$$

and

$$F(x,0) = G(x,\chi(0)) = G(x,0) = f_0(x)\,,$$

it gives one a homotopy between f_0 and f_1.

□

We're now in position to deduce Theorem 1.1.1 from the version of this result that we proved above.

Let

$$\gamma_t : X \times \mathbb{R} \to X \times \mathbb{R}\,, \quad -\infty < t < \infty$$

be the one-parameter group of diffeomorphisms

$$\gamma_t(x,a) = (x, a+t)$$

and let $v = \partial/\partial t$ be the vector field generating this group. For k-forms, $\mu \in \Omega^k(X \times \mathbb{R})$, we have by (1.1.27) the identity

$$\gamma_1^*\mu - \gamma_0^*\mu = d\Gamma\mu + \Gamma\, d\mu \tag{5.1.30}$$

where

$$\Gamma\mu = \int_0^1 \gamma_t^* \left(\iota\left(\frac{\partial}{\partial t}\right)\mu\right)\, dt\,. \tag{5.1.31}$$

Now let F, as in Proposition 1.1.2, be a $\mathcal{C}^\infty$ map

$$F : X \times \mathbb{R} \to Y$$

whose restriction to $X \times [0,1]$ is a homotopy between f_0 and f_1. Then for $\omega \in \Omega^k(Y)$

$$\gamma_1^*F^*\omega - \gamma_0^*F^*\omega = d\Gamma F^*\mu + \Gamma F^*\, d\mu \tag{5.1.32}$$

by the identity (1.1.29). Now let $\iota : X \to X \times \mathbb{R}$ be the inclusion, $p \to (p,0)$, and note that

$$(F \circ \gamma_1 \circ \iota)(p) = F(p,1) = f_1(p)$$

and

$$(F \circ \gamma_0 \circ \iota)(p) = F(p,0) = f_0(p)$$

i.e.,

$$F \circ \gamma_1 \circ \iota \quad = f_1 \tag{5.1.33}$$

and

$$F \circ \gamma_0 \circ \iota \quad = f_0\,. \tag{5.1.34}$$

Thus

$$\iota^*(\gamma_1^* F^* \omega - \gamma_0^* F^* \omega) = f_1^* \omega - f_0^* \omega$$

and on the other hand by (1.1.31)

$$\iota^*(\gamma_1^* F^* \omega - \gamma_0^* F^* \omega) = \, d\iota^* \Gamma F^* \omega + \iota^* \Gamma F^* \, d\omega \, .$$

Letting

$$Q : \Omega^k(Y) \to \Omega^{k-1}(X) \tag{5.1.35}$$

be the "chain homotopy" operator

$$Q\omega = \iota^* \Gamma F^* \omega \tag{5.1.36}$$

we can write the identity above more succinctly in the form

$$f_1^* \omega - f_c^* \omega = \, dQ\omega + Q \, d\omega \tag{5.1.37}$$

and from this deduce, exactly as we did earlier, the identity (1.1.24).

This proof can easily be adapted to the compactly supported setting. Namely the operator (1.1.36) is defined by the integral

$$Q\omega = \int_0^1 \iota^* \gamma_t^* \left(\iota \left(\frac{\partial}{\partial t} \right) F^* \omega \right) \, dt \, . \tag{5.1.38}$$

Hence if ω is supported on a set, A, in Y, the integrand of (1.1.37) at t is supported on the set

$$\{ p \in X \, , \quad F(p,t) \in A \} \tag{5.1.39}$$

and hence $Q\omega$ is supported on the set

$$\pi(F^{-1}(A) \cap X \times [0,1]) \tag{5.1.40}$$

where $\pi : X \times [0,1] \to X$ is the projection map, $\pi(p,t) = p$. Suppose now that f_0 and f_1 are proper mappings and

$$G : X \times [0,1] \to Y$$

a *proper* homotopy between f_0 and f_1, i.e., a homotopy between f_0 and f_1 which is proper as a $\mathcal{C}^\infty$ map. Then if F is the map (1.1.30) its restriction to $X \times [0,1]$ is also a proper map, so this restriction is also a proper homotopy between f_0 and f_1. Hence if ω is in $\Omega_c^k(Y)$ and A is its support, the set (1.1.39) is compact, so $Q\omega$ is in $\Omega_c^{k-1}(X)$. Therefore all summands in the "chain homotopy" formula (1.1.37) are compactly supported. Thus we've proved

Theorem 5.1.3. *If $f_i : X \to Y$, $i = 0, 1$ are proper $\mathcal{C}^\infty$ maps which are homotopic via a proper homotopy, the induced maps on cohomology*

$$f_i^\sharp : H_c^k(Y) \to H_c^k(X)$$

are the same.

We'll conclude this section by noting that the cohomology groups, $H^k(X)$, are equipped with a natural product operation. Namely, suppose $\omega_i \in \Omega^{k_i}(X)$, $i = 1, 2$, is a closed form and that $c_i = [\omega_i]$ is the cohomology class represented by ω_i. We can then define a product cohomology class $c_1 \cdot c_2$ in $H^{k_1+k_2}(X)$ by the recipe

$$c_1 \cdot c_2 = [\omega_1 \wedge \omega_2] \,. \tag{5.1.41}$$

To show that this is a legitimate definition we first note that since ω_2 is closed

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge \, d\omega_2 = 0 \,,$$

so $\omega_1 \wedge \omega_2$ is closed and hence does represent a cohomology class. Moreover if we replace ω_1 by another representative, $\omega_1 + \, d\mu_1 = \omega'$, of the cohomology class, c_1

$$\omega_1' \wedge \omega_2 = \omega_1 \wedge \omega_2 + \, d\mu_1 \wedge \omega_2 \,.$$

But since ω_2 is closed,

$$\begin{aligned} d\mu_1 \wedge \omega_2 &= d(\mu_1 \wedge \omega_2) + (-1)^{k_1} \mu_1 \wedge \, d\omega_2 \\ &= d(\mu_1 \wedge \omega_2) \end{aligned}$$

so

$$\omega_1' \wedge \omega_2 = \omega_1 \wedge \omega_2 + d(\mu_1 \wedge \omega_2)$$

and $[\omega_1' \wedge \omega_2] = [\omega_1 \wedge \omega_2]$. Similary (1.1.41) is unchanged if we replace ω_2 by $\omega_2 + d\mu_2$, so the definition of (1.1.41) depends neither on the choice of ω_1 nor ω_2 and hence is an intrinsic definition as claimed.

There is a variant of this product operation for compactly supported cohomology classes, and we'll leave for you to check that it's also well defined. Suppose c_1 is in $H_c^{k_1}(X)$ and c_2 is in $H^{k_2}(X)$ (i.e., c_1 is a compactly supported class and c_2 is an ordinary cohomology class). Let ω_1 be a representative of c_1 in $\Omega_c^{k_1}(X)$ and ω_2

a representative of c_2 in $\Omega^{k_2}(X)$. Then $\omega_1 \wedge \omega_2$ is a closed form in $\Omega_c^{k_1+k_2}(X)$ and hence defines a cohomology class

$$c_1 \cdot c_2 = [\omega_1 \wedge \omega_2] \tag{5.1.42}$$

in $H_c^{k_1+k_2}(X)$. We'll leave for you to check that this is intrinsically defined. We'll also leave for you to check that (1.1.43) is intrinsically defined if the roles of c_1 and c_2 are reversed, i.e., if c_1 is in $H^{k_1}(X)$ and c_2 in $H_c^{k_2}(X)$ and that the products (1.1.41) and (1.1.43) both satisfy

$$c_1 \cdot c_2 = (-1)^{k_1 k_2} c_2 \cdot c_1 \,. \tag{5.1.43}$$

Finally we note that if Y is another manifold and $f : X \to Y$ a $\mathcal{C}^\infty$ map then for $\omega_1 \in \Omega^{k_1}(Y)$ and $\omega_2 \in \Omega^{k_2}(Y)$

$$f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$$

by (2.5.7) and hence if ω_1 and ω_2 are closed and $c_i = [\omega_i]$

$$f^\sharp(c_1 \cdot c_2) = f^\sharp c_1 \cdot f^\sharp c_2 \,. \tag{5.1.44}$$

Exercises.

1. (Stereographic projection.) Let $p \in S^n$ be the point, $(0, 0, \dots, 0, 1)$. Show that for every point $x = (x_1, \dots, x_{n+1})$ of $S^n - \{p\}$ the ray

$$tx + (1-t)p \,, \quad t > 0$$

intersects the plane, $x_{n+1} = 0$, in the point

$$\gamma(x) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n)$$

and that the map

$$\gamma : S^n - \{p\} \to \mathbb{R}^n \,, \quad x \to \gamma(x)$$

is a diffeomorphism.

2. Show that the operator

$$Q_t : \Omega^k(Y) \to \Omega^{k-1}(X)$$

in the integrand of (1.1.38), i.e., the operator,

$$Q_t\omega = \iota^*\gamma_t^*\left(\iota\left(\frac{\partial}{\partial t}\right)F^*\omega\right)$$

has the following description. Let p be a point of X and let $q = f_t(p)$. The curve, $s \to f_s(p)$ passes through q at time $s = t$. Let $v(q) \in T_qY$ be the tangent vector to this curve at t. Show that

$$(Q_t\omega)(p) = (df_t^*)_p\iota(v_q)\omega_q\,. \tag{5.1.45}$$

3. Let U be a star-shaped open subset of $\mathbb{R}^n$, i.e., a subset of $\mathbb{R}^n$ with the property that for every $p \in U$ the ray, tp, $0 \le t < 1$, is in U.

(a) Let v be the vector field

$$v = \sum x_i \frac{\partial}{\partial x_i}$$

and $\gamma_t : U \to U$, the map $p \to tp$. Show that for every k-form, $\omega \in \Omega^k(U)$

$$\omega = dQ\omega + Q\,d\omega$$

where

$$Q\omega = \int_0^1 \gamma_t^*\iota(v)\omega\frac{dt}{t}\,.$$

(b) Show that if

$$\omega = \sum a_I(x)\,dx_I$$

then

$$Q\omega = \sum_{I,r}\left(\int t^{k-1}(-1)^{r-1}x_{i_r}a_I(tx)\,dt\right)dx_{I_r} \tag{5.1.46}$$

where

$$dx_{I_r} = dx_{i_1}\wedge\cdots\widehat{dx_{i_r}}\wedge\cdots dx_{i_k}\,.$$

4. Let X and Y be oriented connected n-dimensional manifolds, and $f : X \to Y$ a proper map. Show that the linear map, L, in the diagram below

$$\begin{array}{ccc} H^n_c(Y) & \xrightarrow{f^\sharp} & H^n_c(X) \\ {\scriptstyle I_Y}\downarrow & & {\scriptstyle I_X}\downarrow \\ \mathbb{R} & \xrightarrow{L} & \mathbb{R} \end{array}$$

is just the map, $t \in \mathbb{R} \to \deg(f)t$.

5. Let X and Y be manifolds and let id_X and id_Y be the identity maps of X onto X and Y onto Y. A homotopy equivalence between X and Y is a pair of maps

$$f : X \to Y$$

and

$$g : Y \to X$$

such that $g \circ f$ is homotopic to $i\,d_X$ and $f \circ g$ is homotopic to id_Y. Show that if X and Y are homotopy equivalent their cohomology groups are the same "up to isomorphism", i.e., there exist bijections

$$H^k(X) \to H^k(Y)\,.$$

6. Show that $\mathbb{R}^n - \{0\}$ and S^{n-1} are homotopy equivalent.

7. What are the cohomology groups of the n-sphere with two points deleted? *Hint:* The n-sphere with one point deleted is $\mathbb{R}^n$.

8. Let X and Y be manifolds and $f_i : X \to Y$, $i = 0, 1, 2$, $\mathcal{C}^\infty$ maps. Show that if f_0 and f_1 are homotopic and f_1 and f_2 are homotopic then f_0 and f_2 are homotopic.

Hint: The homotopy (1.1.20) has the property that

$$F(p,t) = f_t(p) = f_0(p)$$

for $0 \le t \le \frac{1}{4}$ and

$$F(p,t) = f_t(p) = f_1(p)$$

for $\frac{3}{4} \le t < 1$. Show that two homotopies with these properties: a homotopy between f_0 and f_1 and a homotopy between f_1 and f_2, are easy to "glue together" to get a homotopy between f_0 and f_2.

9. (a) Let X be an n-dimensional manifold. Given points $p_i \in X$, $i = 0, 1, 2$ show that if p_0 can be joined to p_1 by a $\mathcal{C}^\infty$ curve, $\gamma_0 : [0,1] \to X$, and p_1 can be joined to p_2 by a $\mathcal{C}^\infty$ curve, $\gamma_1 : [0,1] \to X$, then p_0 can be joined to p_2 by a $\mathcal{C}^\infty$ curve, $\gamma : [0,1] \to X$.

Hint: A $\mathcal{C}^\infty$ curve, $\gamma : [0,1] \to X$, joining p_0 to p_2 can be thought of as a homotopy between the maps

$$\gamma_{p_0} : pt \to X\,, \quad pt \to p_0$$

and

$$\gamma_{p_1} : pt \to X\,, \quad pt \to p_1$$

where "*pt*" is the zero-dimensional manifold consisting of a single point.

(b) Show that if a manifold, X, is connected it is arc-wise connected: any two points can by joined by a $\mathcal{C}^\infty$ curve.

10. Let X be a connected n-dimensional manifold and $\omega \in \Omega^1(X)$ a closed one-form.

(a) Show that if $\gamma : [0,1] \to X$ is a $\mathcal{C}^\infty$ curve there exists a partition: $0 = a_0 < a_1 < \cdots < a_n = 1$ of the interval $[0,1]$ and open sets U_i in X such that $\gamma\left([a_{i-1}, a_i]\right) \subseteq U_i$ and such that $\omega | U_i$ is exact.

(b) In part (a) show that there exist functions, $f_i \in \mathcal{C}^\infty(U_i)$ such that $\omega | U_i = df_i$ and $f_i\left(\gamma(a_i)\right) = f_{i+1}\left(\gamma(a_i)\right)$.

(c) Show that if p_0 and p_1 are the end points of γ

$$f_n(p_1) - f_1(p_0) = \int_0^1 \gamma^* \omega\,.$$

(d) Let

$$\gamma_s : [0,1] \to X\,, \quad 0 \le s \le 1 \tag{5.1.47}$$

be a homotopic family of curves with $\gamma_s(0) = p_0$ and $\gamma_s(1) = p_1$. Prove that the integral

$$\int_0^1 \gamma_s^* \omega$$

is independent of s_0. *Hint:* Let s_0 be a point on the interval, $[0,1]$. For $\gamma = \gamma_{s_0}$ choose a_i's and f_i's as in parts (a)–(b) and show that for s close to s_0, $\gamma_s[a_{i-1}, a_i] \subseteq U_i$.

(e) A manifold, X, is simply connected if, for any two curves, $\gamma_i : [0,1] \to X$, $i = 0,1$, with the same end-points, p_0 and p_1, there exists a homotopy (1.1.43) with $\gamma_s(0) = p_0$ and $\gamma_s(1) = p_1$, i.e., γ_0 can be smoothly deformed into γ_1 by a family of curves all having the same end-points. Prove

Theorem 5.1.4. *If X is simply-connected $H^1(X) = \{0\}$.*

11. Show that the product operation (1.1.41) is associative and satisfies left and right distributive laws.

12. Let X be a compact oriented $2n$-dimensional manifold. Show that the map

$$B : H^n(X) \times H^n(X) \to \mathbb{R}$$

defined by

$$B(c_1, c_2) = I_X(c_1 \cdot c_2)$$

is a bilinear form on $H^n(X)$ and that it's symmetric if n is even and alternating if n is odd.

5.2 The Mayer–Victoris theorem

In this section we'll develop some techniques for computing cohomology groups of manifolds. (These techniques are known collectively as "diagram chasing" and the mastering of these techniques is more akin to becoming proficient in checkers or chess or the Sunday acrostics in the *New York Times* than in the areas of mathematics to which they're applied.) Let C^i, $i = 0, 1, 2, \ldots$, be vector spaces and $d : C^i \to C^{i+1}$ a linear map. The sequence of vector spaces and maps

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots \tag{5.2.1}$$

is called a *complex* if $d^2 = 0$, i.e., if for $a \in C^k$, $d(da) = 0$. For instance if X is a manifold the DeRham complex

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \cdots \tag{5.2.2}$$

is an example of a complex, and the complex of compactly supported DeRham forms

$$\Omega^0_c(X) \xrightarrow{d} \Omega^1_c(X) \xrightarrow{d} \Omega^2_c(X) \to \cdots \tag{5.2.3}$$

is another example. One defines the *cohomology groups* of the complex (5.2.1) in exactly the same way that we defined the cohomology groups of the complexes (5.2.2) and (5.2.3) in §1.1. Let

$$Z^k = \{a \in C^k \, ; \, da = 0\}$$

and

$$B^k = \{a \in C^k \, ; \, a \in dC^{k-1}\}$$

i.e., let a be in B^k if and only if $a = db$ for some $b \in C^{k-1}$. Then $da = d^2 b = 0$, so B^k is a vector subspace of Z^k, and we define $H^k(C)$ — the k^{th} cohomology group of the complex (5.2.1) — to be the quotient space

(5.2.4) $$H^k(C) = Z^k / B^k \, .$$

Given $c \in Z^k$ we will, as in §1.1, denote its image in $H^k(C)$ by $[c]$ and we'll call c a *representative* of the cohomology class $[c]$.

We will next assemble a small dictionary of "diagram chasing" terms.

Definition 5.2.1. *Let V_i, $i = 0, 1, 2, \ldots$, be vector spaces and $\alpha_i : V_i \to V_{i+1}$ linear maps. The sequence*

(5.2.5) $$V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots$$

is an exact sequence *if, for each i, the kernel of α_{i+1} is equal to the image of α_i.*

For example the sequence (5.2.1) is exact if $Z_i = B_i$ for all i, or, in other words, if $H^i(C) = 0$ for all i. A simple example of an exact sequence that we'll encounter a lot below is a sequence of the form

(5.2.6) $$\{0\} \to V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} V_3 \to \{0\} \, ,$$

a five term exact sequence whose first and last terms are the vector space, $V_0 = V_4 = \{0\}$, and hence $\alpha_0 = \alpha_3 = 0$. This sequence is exact if and only if

1. α_1 is injective,

2. the kernel of α_2 equals the image of α_1, and

3. α_2 is surjective.

We will call an exact sequence of this form a *short exact* sequence. (We'll also encounter a lot below an even shorter example of an exact sequence, namely a sequence of the form

$$\{0\} \to V_1 \xrightarrow{\alpha_1} V_2 \to \{0\}\,. \tag{5.2.7}$$

This is an exact sequence if and only if α_1 is bijective.)

Another basic notion in the theory of diagram chasing is the notion of a commutative diagram. The square diagram of vector spaces and linear maps

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ C \arrow[u, "i"] \arrow[r, "g"] & D \arrow[u, "j"'] \end{tikzcd}$$

is commutative if $f \circ i = j \circ g$, and a more complicated diagram of vector spaces and linear maps like the diagram below

$$\begin{tikzcd} A_1 \arrow[r] & A_2 \arrow[r] & A_3 \\ B_1 \arrow[u] \arrow[r] & B_2 \arrow[u] \arrow[r] & B_3 \arrow[u] \\ C_1 \arrow[u] \arrow[r] & C_2 \arrow[u] \arrow[r] & C_3 \arrow[u] \end{tikzcd}$$

is commutative if every subsquare in the diagram, for instance the square,

$$\begin{tikzcd} B_2 \arrow[r] & B_3 \\ C_2 \arrow[u] \arrow[r] & C_3 \arrow[u] \end{tikzcd}$$

is commutative.

We now have enough "diagram chasing" vocabulary to formulate the Mayer–Victoris theorem. For $r = 1, 2, 3$ let

$$\{0\} \to C_r^0 \xrightarrow{d} C_r^1 \xrightarrow{d} C_r^2 \xrightarrow{d} \cdots \tag{5.2.8}$$

be a complex and, for fixed k, let

$$\{0\} \to C_1^k \xrightarrow{i} C_2^k \xrightarrow{j} C_3^k \to \{0\} \tag{5.2.9}$$

be a short exact sequence. Assume that the diagram below commutes:

$$
\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & C_1^{k+1} \arrow[u, no head] \arrow[r, "i"] & C_2^{k+1} \arrow[u, no head] \arrow[r, "j"] & C_3^{k+1} \arrow[u, no head] \arrow[r] & 0 \\
0 \arrow[r] & C_1^{k} \arrow[u, "d"] \arrow[r] & C_2^{k} \arrow[u, "d"] \arrow[r] & C_3^{k} \arrow[u, "d"] \arrow[r] & 0 \\
0 \arrow[r] & C_1^{k-1} \arrow[u] \arrow[r, "i"] & C_2^{k-1} \arrow[u] \arrow[r, "j"] & C_3^{k-1} \arrow[u] \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
\tag{5.2.10}
$$

i.e., assume that in the left hand squares, $di = id$, and in the right hand squares, $dj = jd$.

The Mayer–Victoris theorem addresses the following question: If one has information about the cohomology groups of two of the three complexes, (5.2.8), what information about the cohomology groups of the third can be extracted from this diagram? Let's first observe that the maps, i and j, give rise to mappings between these cohomology groups. Namely, for $r = 1, 2, 3$ let Z_r^k be the kernel of the map, $d : C_r^k \to C_r^{k+1}$, and B_r^k the image of the map, $d : C_r^{k-1} \to C_r^k$. Since $id = di$, i maps B_1^k into B_2^k and Z_1^k into Z_2^k, therefore by (5.2.4) it gives rise to a linear mapping

$$i_\sharp : H^k(C_1) \to H^k(C_2)\,.$$

Similarly since $jd = dj$, j maps B_2^k into B_3^k and Z_2^k into Z_3^k, and so by (5.2.4) gives rise to a linear mapping

$$j_\sharp : H^k(C_2) \to H^k(C_3)\,.$$

Moreover, since $j \circ i = 0$ the image of $i_\sharp$ is contained in the kernel of $j_\sharp$. We'll leave as an exercise the following sharpened version of this observation:

Proposition 5.2.2. *The kernel of $j_\sharp$ equals the image of $i_\sharp$, i.e., the three term sequence*

$$H^k(C_1) \xrightarrow{i_\sharp} H^k(C_2) \xrightarrow{j_\sharp} H^k(C_3) \tag{5.2.11}$$

is exact.

Since (5.2.9) is a short exact sequence one is tempted to conjecture that (5.2.11) is also a short exact sequence (which, if it were true, would tell us that the cohomology groups of any two of the complexes (5.2.8) completely determine the cohomology groups of the third). Unfortunately, this is not the case. To see how this conjecture can be violated let's try to show that the mapping $j_\sharp$ is surjective. Let c_3^k be an element of Z_3^k representing the cohomology class, $[c_3^k]$, in $H^3(C_3)$. Since (5.2.9) is exact there exists a c_2^k in C_2^k which gets mapped by j onto c_3^k, and if c_3^k were in Z_2^k this would imply

$$j_\sharp[c_2^k] = [jc_2^k] = [c_3^k]\,,$$

i.e., the cohomology class, $[c_3^k]$, would be in the image of $j_\sharp$. However, since there's no reason for c_2^k to be in Z_2^k, there's also no reason for $[c_3^k]$ to be in the image of $j_\sharp$. What we can say, however, is that $j\,dc_2^k = djc_2^k = dc_3^k = 0$ since c_3^k is in Z_3^k. Therefore by the exactness of (5.2.9) in degree $k+1$ there exists a unique element, c_1^{k+1} in C_1^{k+1} with property

$$dc_2^k = ic_1^{k+1}\,. \tag{5.2.12}$$

Moreover, since $0 = d(dc_2^k) = di(c_1^{k+1}) = i\,dc_1^{k+1}$ and i is injective, $dc_1^{k+1} = 0$, i.e.,

$$c_1^{k+1} \in Z_1^{k+1}\,. \tag{5.2.13}$$

Thus via (5.2.12) and (5.2.13) we've converted an element, c_3^k, of Z_3^k into an element, c_1^{k+1}, of Z_1^{k+1} and hence set up a correspondence

$$c_3^k \in Z_3^k \to c_1^{k+1} \in Z_1^{k+1}\,. \tag{5.2.14}$$

Unfortunately this correspondence isn't, strictly speaking, a map of Z_3^k into Z_1^{k+1}; the c_1^k in (5.2.14) isn't determined by c_3^k alone but also by the choice we made of c_2^k. Suppose, however, that we make another choice of a c_2^k with the property $j(c_2^k) = c_3^k$. Then the difference between our two choices is in the kernel of j and hence, by the exactness of (2.5.8) at level k, is in the image of i. In other words, our two choices are related by

$$(c_2^k)_{\text{new}} = (c_2^k)_{old} + i(c_1^k)$$

for some c_1^k in C_1^k, and hence by (5.2.12)

$$(c_1^{k+1})_{\text{new}} = (c_1^{k+1})_{old} + dc_1^k\,.$$

Therefore, even though the correspondence (5.2.14) isn't strictly speaking a map it does give rise to a well-defined map

$$Z_3^k \to H^{k+1}(C_1)\,, \quad c_3^k \to [c_3^{k+1}]\,. \tag{5.2.15}$$

Moreover, if c_3^k is in B_3^k, i.e., $c_3^k = dc_3^{k-1}$ for some $c_3^{k-1} \in C_3^{k-1}$, then by the exactness of (5.2.8) at level $k-1$, $c_3^{k-1} = j(c_2^{k-1})$ for some $c_2^{k-1} \in C_2^{k-1}$ and hence $c_3^k = j(dc_2^{k-2})$. In other words we can take the c_2^k above to be dc_2^{k-1} in which case the c_1^{k+1} in equation (5.2.12) is just zero. Thus the map (5.2.14) maps B_3^k to zero and hence by Proposition 1.2.2 gives rise to a well-defined map

$$\delta : H^k(C_3) \to H^{k+1}(C_1) \tag{5.2.16}$$

mapping $[c_3^k] \to [c_1^{k+1}]$. We will leave it as an exercise to show that this mapping measures the failure of the arrow $j_\sharp$ in the exact sequence (5.2.11) to be surjective (and hence the failure of this sequence to be a short exact sequence at its right end).

Proposition 5.2.3. *The image of the map* $j_\sharp : H^k(C_2) \to H^k(C_3)$ *is equal to the kernel of the map,* $\delta : H^k(C_3) \to H^{k+1}(C_1)$.

Hint: Suppose that in the correspondence (5.2.14) c_1^{k+1} is in B_1^{k+1}. Then $c_1^{k+1} = dc_1^k$ for some c_1^k in C_1^k. Show that

$$j(c_2^k - i(c_1^k)) = c_3^k$$

and

$$d(c_2^k - i(c_1^k)) = 0$$

i.e., $c_2^k - i(c_1^k)$ is in Z_2^k and hence $j_\sharp[c_2^k - i(c_1^k)] = [c_3^k]$.

Let's next explore the failure of the map, $i_\sharp : H^{k+1}(C_1) \to H^{k+1}(C_2)$, to be injective. Let c_1^{k+1} be in Z_1^{k+1} and suppose that its cohomology class, $[c_1^{k+1}]$, gets mapped by $i_\sharp$ into zero. This translates into the statement

$$i(c_1^{k+1}) = dc_2^k \tag{5.2.17}$$

for some $c_2^k \in C_2^k$. Moreover since $dc_2^k = i(c_1^{k+1})$, $j(dc_2^k) = 0$. But if

$$c_3^k \stackrel{\text{def}}{=} j(c_2^k) \tag{5.2.18}$$

then $dc_3^k = dj(c_2^k) = j(dc_2^k) = j(i(c_1^{k+1})) = 0$, so c_3^k is in Z_3^k, and by (5.2.17), (5.2.18) and the definition of δ

$$[c_1^{k+1}] = \delta[c_3^k]\,. \tag{5.2.19}$$

In other words the kernel of the map, $i_\sharp : H^{k+1}(C_1) \to H^{k+1}(C_2)$ is contained in the image of the map $\delta : H^k(C_3) \to H^{k+1}(C_1)$. We will leave it as an exercise to show that this argument can be reversed to prove the converse assertion and hence to prove

Proposition 5.2.4. *The image of the map* $\delta : H^k(C_1) \to H^{k+1}(C_1)$ *is equal to the kernel of the map* $i_\sharp : H^{k+1}(C_1) \to H^{k+1}(C_2)$.

Putting together the Propositions 5.2.2–5.2.4 we obtain the main result of this section: the Mayer–Victoris theorem. The sequence of cohomology groups and linear maps

$$\cdots \xrightarrow{\delta} H^k(C_1) \xrightarrow{i_\sharp} H^k(C_2) \xrightarrow{j_\sharp} H^k(C_3) \xrightarrow{\delta} H^{k+1}(C-1) \xrightarrow{i_\sharp} \cdots \tag{5.2.20}$$

is exact.

Remark 5.2.5. *In view of the "$\cdots$"'s this sequence can be a* very *long sequence and is commonly referred to as the "long exact sequence in cohomology" associated to the short exact sequence of complexes (2.5.9).*

Before we discuss the applications of this result, we will introduce some vector space notation. Given vector spaces, V_1 and V_2 we'll denote by $V_1 \oplus V_2$ the vector space sum of V_1 and V_2, i.e., the set of all pairs

$$(u_1, u_2)\,, \quad u_i \in V_i$$

with the addition operation

$$(u_1, u_2) + (\mathrm{v}_1 + \mathrm{v}_2) = (u_1 + \mathrm{v}_1\,,\ u_2 + \mathrm{v}_2)$$

and the scalar multiplication operation

$$\lambda(u_1, u_2) = (\lambda u_1\,,\ \lambda u_2)\,.$$

Now let X be a manifold and let U_1 and U_2 be open subsets of X. Then one has a linear map

$$\Omega^k(U_1 \cup U_2) \xrightarrow{i} \Omega^k(U_1) \oplus \Omega^k(U_2) \tag{5.2.21}$$

defined by

$$\omega \to (\omega|U_1\,,\, \omega|U_2) \tag{5.2.22}$$

where $\omega|U_i$ is the restriction of ω to U_i. Similarly one has a linear map

$$\Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{j} \Omega^k(U_1 \cap U_2) \tag{5.2.23}$$

defined by

$$(\omega_1, \omega_2) \to \omega_1|U_1 \cap U_2 - \omega_2|U_1 \cap U_2\,. \tag{5.2.24}$$

We claim

Theorem 5.2.6. *The sequence*

(5.2.25)

$$\{0\} \to \Omega^k(U_1 \cup U_2) \xrightarrow{i} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{j} \Omega^k(U_1 \cap U_2) \to \{0\}$$

is a short exact sequence.

Proof. If the right hand side of (5.2.22) is zero, ω itself has to be zero so the map (5.2.22) is injective. Moreover, if the right hand side of (5.2.24) is zero, ω_1 and ω_2 are equal on the overlap, $U_1 \cap U_2$, so we can glue them together to get a $\mathcal{C}^\infty$ k-form on $U_1 \cup U_2$ by setting $\omega = \omega_1$ on U_1 and $\omega = \omega_2$ on U_2. Thus by (5.2.22) $i(\omega) = (\omega_1, \omega_2)$, and this shows that the kernel of j is equal to the image of i. Hence to complete the proof we only have to show that j is surjective, i.e., that every form ω on $\Omega^k(U_1 \cap U_2)$ can be written as a difference, $\omega_1|U_1 \cap U_2 - \omega_2|U_1 \cap U_2$, where ω_1 is in $\Omega^k(U_1)$ and ω_2 in in $\Omega^k(U_2)$. To prove this we'll need the following variant of the partition of unity theorem.

Theorem 5.2.7. *There exist functions, $\varphi_\alpha \in \mathcal{C}^\infty(U_1 \cup U_2)$, $\alpha = 1, 2$, such that support φ_α is contained in U_α and $\varphi_1 + \varphi_2 = 1$.*

Before proving this let's use it to complete our proof of Theorem 5.2.6. Given $\omega \in \Omega^k(U_1 \cap U_2)$ let

$$\omega_1 = \begin{cases} \varphi_2\omega & \text{on} \quad U_1 \cap U_2 \\ 0 & \text{on} \quad U_1 - U_1 \cap U_2 \end{cases} \tag{5.2.26}$$

and let

$$\omega_2 = \begin{cases} -\varphi_1\omega & \text{on} \quad U_1 \cap U_2 \\ 0 & \text{on} \quad U_2 - U_1 \cap U_2 \,. \end{cases} \tag{5.2.27}$$

Since φ_2 is supported on U_2 the form defined by (5.2.26) is $\mathcal{C}^\infty$ on U_1 and since φ_1 is supported on U_1 the form defined by (5.2.27) is $\mathcal{C}^\infty$ on U_2 and since $\varphi_1 + \varphi_2 = 1$, $\omega_1 - \omega_2 = (\varphi_1 + \varphi_2)\omega = \omega$ on $U_1 \cap U_2$. □

To prove Theorem 5.2.7, let $\rho_i \in \mathcal{C}_0^\infty(U_1 \cup U_2)$, $i = 1, 2, 3, \ldots$ be a partition of unity subordinate to the cover, $\{U_\alpha\,,\ \alpha = 1, 2\}$ of $U_1 \cup U_2$ and let φ_1 be the sum of the ρ_i's with support on U_1 and φ_2 the sum of the remaining ρ_i's. It's easy to check (using part (b) of Theorem 4.6.1) that φ_α is supported in U_α and (using part (c) of Theorem 4.6.1) that $\varphi_1 + \varphi_2 = 1$. □

Now let

$$\{0\} \to C_1^0 \xrightarrow{d} C_1^1 \xrightarrow{d} C_1^2 \to \cdots \tag{5.2.28}$$

be the DeRham complex of $U_1 \cup U_2$, let

$$\{0\} \to C_3^0 \xrightarrow{d} C_3^1 \xrightarrow{d} C_3^2 \to \cdots \tag{5.2.29}$$

be the DeRham complex of $U_1 \cap U_2$ and let

$$\{0\} \to C_2^0 \xrightarrow{d} C_2^1 \xrightarrow{d} C_2^2 \xrightarrow{d} \cdots \tag{5.2.30}$$

be the vector space direct sum of the DeRham complexes of U_1 and U_2, i.e., the complex whose k^{th} term is

$$C_2^k = \Omega^k(U_1) \oplus \Omega^k(U_2)$$

with $d : C_2^k \to C_2^{k+1}$ defined to be the map $d(\mu_1, \mu_2) = (d\mu_1, d\mu_2)$. Since $C_1^k = \Omega^k(U_1 \cup U_2)$ and $C_3^k = \Omega^k(U_1 \cap U_2)$ we have, by Theorem 5.2.6, a short exact sequence

$$\{0\} \to C_1^k \xrightarrow{i} C_2^k \xrightarrow{j} C_3^k \to \{0\}\,, \tag{5.2.31}$$

and it's easy to see that i and j commute with the d's:

$$di = id \text{ and } dj = jd\,. \tag{5.2.32}$$

Hence we're exactly in the situation to which Mayer–Victoris applies. Since the cohomology groups of the complexes (5.2.28) and (5.2.29) are the DeRham cohomology group. $H^k(U_1 \cup U_2)$ and $H^k(U_1 \cap U_2)$, and the cohomology groups of the complex (5.2.30) are the vector space direct sums, $H^k(U_1) \oplus H^k(U_2)$, we obtain from the abstract Mayer–Victoris theorem, the following DeRham theoretic version of Mayer–Victoris.

Theorem 5.2.8. *Letting $U = U_1 \cup U_2$ and $V = U_1 \cap U_2$ one has a long exact sequence in DeRham cohomology:*

(5.2.33)
$$\cdots \xrightarrow{\delta} H^k(U) \xrightarrow{i_\sharp} H^k(U_1) \oplus H^k(U_2) \xrightarrow{j_\sharp} H^k(V) \xrightarrow{\delta} H^{k+1}(U) \xrightarrow{i_\sharp} \cdots .$$

This result also has an analogue for compactly supported DeRham cohomology. Let

(5.2.34)
$$i : \Omega_c^k(U_1 \cap U_2) \to H_c^k(U_1) \oplus \Omega_c^k(U_2)$$

be the map

(5.2.35)
$$i(\omega) = (\omega_1, \omega_2)$$

where

(5.2.36)
$$\omega_i = \begin{cases} \omega & \text{on} \quad U_1 \cap U_2 \\ 0 & \text{on} \quad U_i - U_1 \cap U_2 \,. \end{cases}$$

(Since ω is compactly supported on $U_1 \cap U_2$ the form defined by (5.2.34) is a $\mathcal{C}^\infty$ form and is compactly supported on U_i.) Similarly, let

(5.2.37)
$$j : \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \to \Omega_c^k(U_1 \cup U_2)$$

be the map

(5.2.38)
$$j(\omega_1, \omega_2) = \widetilde{\omega}_1 - \widetilde{\omega}_2$$

where:

(5.2.39)
$$\widetilde{\omega}_i = \begin{cases} \omega_i & \text{on} \quad U_i \\ 0 & \text{on} \quad (U_1 \cup U_2) - U_i \,. \end{cases}$$

As above it's easy to see that i is injective and that the kernel of j is equal to the image of i. Thus if we can prove that j is surjective we'll have proved

Theorem 5.2.9. *The sequence*

(5.2.40)
$$\{0\} \to \Omega_c^k(U_1 \cap U_2) \xrightarrow{i} \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \xrightarrow{j} \Omega_c^k(U_1 \cap U_2) \to \{0\}$$

is a short exact sequence.

Proof. To prove the surjectivity of j we mimic the proof above. Given ω in $\Omega_c^k(U_1 \cup U_2)$ let

(5.2.41)
$$\omega = \varphi_1 \omega | U_1$$

and

(5.2.42)
$$\omega_2 = -\varphi_2 \omega | U_2 \,.$$

Then by (5.2.36) $\omega = j(\omega_1, \omega_2)$.

□

Thus, applying Mayer–Victoris to the compactly supported versions of the complexes (5.2.8), we obtain:

Theorem 5.2.10. *Letting $U = U_1 \cup U_2$ and $V = U_1 \cap U_2$ there exists a long exact sequence in compactly supported DeRham cohomology*

(5.2.43)
$$\cdots \xrightarrow{\delta} H_c^k(V) \xrightarrow{i_\sharp} H_c^k(U_1) \oplus H_c^k(U_2) \xrightarrow{j_\sharp} H_c^k(U) \xrightarrow{\delta} H_c^{k+1}(V) \xrightarrow{i_\sharp} \cdots \,.$$

Exercises

1. Prove Proposition 5.2.2.

2. Prove Proposition 5.2.3.

3. Prove Proposition 5.2.4.

4. Show that if U_1, U_2 and $U_1 \cap U_2$ are non-empty and connected the first segment of the Mayer–Victoris sequence is a short exact sequence

$$\{0\} \to H^0(U_1 \cup U_2) \to H^0(U_1) \oplus H^0(U_2) \to H^0(U_1 \cap U_2) \to \{0\} \,.$$

5. Let $X = S^n$ and let U_1 and U_2 be the open subsets of S^n obtained by removing from S^n the points, $p_1 = (0, \ldots, 0, 1)$ and $p_2 = (0, \ldots, 0, -1)$.

(a) Using stereographic projection show that U_1 and U_2 are diffeomorphic to $\mathbb{R}^n$.

(b) Show that $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is homotopy equivalent to S^{n-1}. (See problem 5 in §1.1.) *Hint:* $U_1 \cap U_2$ is diffeomorphic to $\mathbb{R}^n - \{0\}$.

(c) Deduce from the Mayer–Victoris sequence that $H^{i+1}(S^n) = H^i(S^{n-1})$ for $i \geq 1$.

(d) Using part (c) give an inductive proof of a result that we proved by other means in §1.1: $H^k(S^n) = \{0\}$ for $1 \leq k < n$.

6. Using the Mayer–Victoris sequence of exercise 5 with cohomology replaced by compactly supported cohomology show that

$$H_c^k(\mathbb{R}^n - \{0\}) \quad \cong \mathbb{R}$$

for $k = 1$ and n and

$$H_c^k(\mathbb{R}^n - \{0\}) \quad = \{0\}$$

for all other values of k.

5.3 Good covers

In this section we will show that for compact manifolds (and for lots of other manifolds besides) the DeRham cohomology groups which we defined in §1.1 are finite dimensional vector spaces and thus, in principle, "computable" objects. A key ingredient in our proof of this fact is the notion of a *good cover* of a manifold.

Definition 5.3.1. *Let X be an n-dimensional manifold, and let*

$$\mathbb{U} = \{U_\alpha, \alpha \in \mathcal{I}\}$$

be a covering of X by open sets. This cover is a good *cover if for every finite set of indices, $\alpha_i \in \mathcal{I}$, $i = 1, \ldots, k$, the intersection $U\alpha_1 \cap \cdots \cap U\alpha_k$ is either empty or is diffeomorphic to $\mathbb{R}^n$.*

One of our first goals in this section will be to show that good covers exist. We will sketch below a proof of the following.

Theorem 5.3.2. *Every manifold admits a good cover.*

The proof involves an elementary result about open convex subsets of $\mathbb{R}^n$.

Proposition 5.3.3. *If U is a bounded open convex subset of $\mathbb{R}^n$, it is diffeomorphic to $\mathbb{R}^n$.*

A proof of this will be sketched in exercises 1–4 at the end of this section.

One immediate consequence of this result is an important special case of Theorem 5.3.2.

Theorem 5.3.4. *Every open subset, U, of $\mathbb{R}^n$ admits a good cover.*

Proof. For each $p \in U$ let U_p be an open convex neighborhood of p in U (for instance an ϵ-ball centered at p) . Since the intersection of any two convex sets is again convex the cover, $\{U_p, p \in U\}$ is a good cover by Proposition 5.3.3.

□

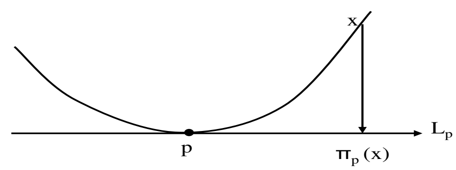
For manifolds the proof of Theorem 5.3.2 is somewhat trickier. The proof requires a manifold analogue of the notion of convexity and there are several serviceable candidates. The one we will use is the following. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold and for $p \in X$ let T_pX be the tangent space to X at p. Recalling that T_pX sits inside $T_p\mathbb{R}^N$ and that

$$T_p\mathbb{R}^N = \{(p, \mathrm{v}) \, , \, \mathrm{v} \in \mathbb{R}^N\}$$

we get a map

$$T_pX \hookrightarrow T_p\mathbb{R}^N \to \mathbb{R}^N \, , \quad (p, x) \to p + x \, ,$$

and this map maps T_pX bijectively onto an n-dimensional "affine" subspace, L_p, of $\mathbb{R}^N$ which is tangent to X at p. Let $\pi_p : X \to L_p$ be, as in the figure below, the orthogonal projection of X onto L_p.



Definition 5.3.5. *An open subset, V, of X is* convex *if for every $p \in V$ the map $\pi_p : X \to L_p$ maps V diffeomorphically onto a convex open subset of L_p.*

It's clear from this definition of convexity that the intersection of two open convex subsets of X is an open convex subset of X and that every open convex subset of X is diffeomorphic to $\mathbb{R}^n$. Hence to prove Theorem 5.3.2 it suffices to prove that every point, p, in X is contained in an open convex subset, U_p, of X. Here is a sketch of how to prove this. In the figure above let $B^\epsilon(p)$ be the ball of radius ϵ about p in L_p centered at p. Since L_p and T_p are tangent at p the derivative of π_p at p is just the identity map, so for ϵ small π_p maps a neighborhood, U_p^ϵ of p in X diffeomorphically onto $B^\epsilon(p)$. We claim

Proposition 5.3.6. *For ϵ small, U_p^ϵ is a convex subset of X.*

Intuitively this assertion is pretty obvious: if q is in U_p^ϵ and ϵ is small the map

$$B_p^\epsilon \xrightarrow{\pi_p^{-1}} U_p^\epsilon \xrightarrow{\pi_q} L_q$$

is to order ϵ^2 equal to the identity map, so it's intuitively clear that its image is a slightly warped, but still convex, copy of $B^\epsilon(p)$. We won't, however, bother to write out the details that are required to make this proof rigorous.

A good cover is a particularly good "good cover" if it is a finite cover. We'll codify this property in the definition below.

Definition 5.3.7. *An n-dimensional manifold is said to have* finite topology *if it admits a finite covering by open sets, $U_1, \dots, U_N$ with the property that for every multi-index, $I = (i_1, \dots, i_k)$, $1 \le i_1 \le i_2 \cdots < i_K \le N$, the set*

$$U_I = Ui_1 \cap \cdots \cap Ui_k \tag{5.3.1}$$

is either empty or is diffeomorphic to $\mathbb{R}^n$.

If X is a compact manifold and $\mathbb{U} = \{U_\alpha\,,\ \alpha \in \mathcal{I}\}$ is a good cover of X then by the Heine–Borel theorem we can extract from $\mathbb{U}$ a finite subcover

$$U_i = U_{\alpha_i}\,,\ \alpha_i \in \mathcal{I}\,,\ i = 1, \dots, N\,,$$

hence we conclude

Theorem 5.3.8. *Every compact manifold has finite topology.*

More generally, for any manifold, X, let C be a compact subset of X. Then by Heine–Borel we can extract from the cover, $\mathbb{U}$, a finite subcollection

$$U_i = U_{\alpha_i}\,, \quad \alpha_i \in \mathcal{I}\,, \quad i = 1, \dots, N$$

that covers C, hence letting $U = \bigcup U_i$, we've proved

Theorem 5.3.9. *If X is an n-dimensional manifold and C a compact subset of X, then there exists an open neighborhood, U, of C in X having finite topology.*

We can in fact even strengthen this further. Let U_0 be any open neighborhood of C in X. Then in the theorem above we can replace X by U_0 to conclude

Theorem 5.3.10. *Let X be a manifold, C a compact subset of X and U_0 an open neighborhood of C in X. Then there exists an open neighborhood, U, of C in X, U contained in U_0, having finite topology.*

We will justify the term "finite topology" by devoting the rest of this section to proving

Theorem 5.3.11. *Let X be an n-dimensional manifold. If X has finite topology the DeRham cohomology groups, $H^k(X)$, $k = 0, \dots, n$ and the compactly supported DeRham cohomology groups, $H^k_c(X)$, $k = 0, \dots, n$ are finite dimensional vector spaces.*

The basic ingredients in the proof of this will be the Mayer–Victoris techniques that we developed in §5.2 and the following elementary result about vector spaces.

Lemma 5.3.12. *Let V_i, $i = 1, 2, 3$, be vector spaces and*

$$V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \tag{5.3.2}$$

an exact sequence of linear maps. Then if V_1 and V_3 are finite dimensional, so is V_2.

Proof. Since V_3 is finite dimensional, the image of β is of dimension, $k < \infty$, so there exist vectors, v_i, $i = 1, \dots, k$ in V_2 having the property that

$$\operatorname{Image} \beta = \operatorname{span} \{\beta(\mathrm{v}_i) , \quad i = 1, \dots, k\}. \tag{5.3.3}$$

Now let v be any vector in V_2. Then $\beta(\mathrm{v})$ is a linear combination

$$\beta(\mathrm{v}) = \sum_{i=1}^{k} c_i \beta(\mathrm{v}_i) \quad c_i \in \mathbb{R}$$

of the vectors $\beta(\mathrm{v}_i)$ by (5.3.3), so

$$\mathrm{v}' = \mathrm{v} - \sum_{i=1}^{k} c_i \mathrm{v}_i \tag{5.3.4}$$

is in the kernel of β and hence, by the exactness of (5.3.2), in the image of α. But V_1 is finite dimensional, so $\alpha(V_1)$ is finite dimensional. Letting $\mathrm{v}_{k+1}, \dots, \mathrm{v}_m$ be a basis of $\alpha(V_1)$ we can by (5.3.4) write v as a sum, $\mathrm{v} = \sum_{i=1}^{m} c_i \mathrm{v}_i$. In other words $\mathrm{v}_1, \dots, \mathrm{v}_m$ is a basis of V_2. □

We'll now prove Theorem 5.3.4. Our proof will be by induction on the number of open sets in a good cover of X. More specifically let

$$\mathbb{U} = \{U_i \, , \, i = 1, \dots, N\}$$

be a good cover of X. If $N = 1$, $X = U_1$ and hence X is diffeomorphic to $\mathbb{R}^n$, so

$$H^k(X) = \{0\} \text{ for } k > 0$$

and $H^k(X) = \mathbb{R}$ for $k = 0$, so the theorem is certainly true in this case. Let's now prove it's true for arbitrary N by induction. Let U be the open subset of X obtained by forming the union of $U_2, \dots, U_N$. We can think of U as a manifold in its own right, and since $\{U_i\, , \, i = 2, \dots, N\}$ is a good cover of U involving only $N - 1$ sets, its cohomology groups are finite dimensional by the induction assumption. The same is also true of the intersection of U with U_1. It has the $N - 1$ sets, $U \cap U_i$, $i = 2, \dots, N$ as a good cover, so its cohomology groups are finite dimensional as well. To prove that the theorem is true for X we note that $X = U_1 \cup U$ and that one has an exact sequence

$$H^{k-1}(U_1 \cap U) \xrightarrow{\delta} H^k(X) \xrightarrow{i_\sharp} H^k(U_1) \oplus H^k(U)$$

by Mayer–Victoris. Since the right hand and left hand terms are finite dimensional it follows from Lemma 5.3.12 that the middle term is also finite dimensional. □

The proof works practically verbatim for compactly supported cohomology. For $N = 1$

$$H_c^k(X) = H_c^k(U_1) = H_c^k(\mathbb{R}^n)$$

so all the cohomology groups of $H^k(X)$ are finite in this case, and the induction "$N - 1$" ⇒ "N" follows from the exact sequence

$$H_c^k(U_1) \oplus H_c^k(U) \xrightarrow{j_\sharp} H_c^k(X) \xrightarrow{\delta} H_c^{k+1}(U_1 \cap U)\,.$$

Remark 5.3.13. *A careful analysis of the proof above shows that the dimensions of the $H^k(X)$'s are determined by the intersection properties of the U_i's, i.e., by the list of multi-indices, I, for which th intersections (5.3.1) are non-empty.*

This collection of multi-indices is called the *nerve* of the cover, $\mathbb{U} = \{U_i\, , \, i = 1, \dots, N\}$, and this remark suggests that there should be a cohomology theory which has as input the nerve of $\mathbb{U}$ and as output cohomology groups which are isomorphic to the DeRham cohomology groups. Such a theory does exist and a nice account of it can be found in Frank Warner's book, "Foundations of Differentiable Manifolds and Lie Groups". (See the section on Čech cohomology in Chapter 5.)

Exercises.

1. Let U be a bounded open subset of $\mathbb{R}^n$. A continuous function

$$\psi : U \to [0, \infty)$$

is called an exhaustion function if it is proper as a map of U into $[0, \infty)$; i.e., if, for every $a > 0$, $\psi^{-1}([0, a])$ is compact. For $x \in U$ let

$$d(x) = \inf \{|x - y| , \quad y \in \mathbb{R}^n - U\} ,$$

i.e., let $d(x)$ be the "distance" from x to the boundary of U. Show that $d(x) > 0$ and that $d(x)$ is continuous as a function of x. Conclude that $\psi_0 = 1/d$ is an exhaustion function.

2. Show that there exists a $\mathcal{C}^\infty$ exhaustion function, $\varphi_0 : U \to [0, \infty)$, with the property $\varphi_0 \geq \psi_0^2$ where ψ_0 is the exhaustion function in exercise 1.

Hints: For $i = 2, 3, \ldots$ let

$$C_i = \left\{ x \in U , \quad \frac{1}{i} \leq d(x) \leq \frac{1}{i-1} \right\}$$

and

$$U_i = \left\{ x \in U , \quad \frac{1}{i+1} < d(x) < \frac{1}{i-2} \right\} .$$

Let $\rho_i \in \mathcal{C}_0^\infty(U_i)$, $\rho_i \geq 0$, be a "bump" function which is identically one on C_i and let $\varphi_0 = \sum i^2 \rho_i + 1$.

3. Let U be a bounded open convex subset of $\mathbb{R}^n$ containing the origin. Show that there exists an exhaustion function

$$\psi : U \to \mathbb{R} , \quad \psi(0) = 1 ,$$

having the property that ψ is a *monotonically increasing* function of t along the ray, tx, $0 \leq t \leq 1$, for all points, x, in U. *Hints:*

(a) Let $\rho(x)$, $0 \leq \rho(x) \leq 1$, be a $\mathcal{C}^\infty$ function which is one outside a small neighborhood of the origin in U and is zero in a still smaller neighborhood of the origin. Modify the function, φ_0, in the previous exercise by setting $\varphi(x) = \rho(x)\varphi_0(x)$ and let

$$\psi(x) = \int_0^1 \varphi(sx)\frac{ds}{s} + 1\,.$$

Show that for $0 \le t \le 1$

$$\frac{d\psi}{dt}(tx) = \varphi(tx)/t \tag{5.3.5}$$

and conclude from (5.3.4) that ψ is monotonically increasing along the ray, tx, $0 \le t \le 1$.

(b) Show that for $0 < \epsilon < 1$,

$$\psi(x) \ge \epsilon\varphi(y)$$

where y is a point on the ray, tx, $0 \le t \le 1$ a distance less than $\epsilon|x|$ from X.

(c) Show that there exist constants, C_0 and C_1, $C_1 > 0$ such that

$$\psi(x) = \frac{C_1}{d(x)} + C_0\,.$$

Sub-hint: In part (b) take ϵ to be equal to $\frac{1}{2}\,d(x)/|x|$.

4. Show that every bounded, open convex subset, U, of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. *Hints:*

(a) Let $\psi(x)$ be the exhaustion function constructed in exercise 3 and let

$$f : U \to \mathbb{R}^n$$

be the map: $f(x) = \psi(x)x$. Show that this map is a bijective map of U onto $\mathbb{R}^n$.

(b) Show that for $x \in U$ and $\mathrm{v} \in \mathbb{R}^n$

$$(df)_x\mathrm{v} = \psi(x)\mathrm{v} + d\psi_x(\mathrm{v})x$$

and conclude that df_x is bijective at x, i.e., that f is locally a diffeomorphism of a neighborhood of x in U onto a neighborhood of $f(x)$ in $\mathbb{R}^n$.

(c) Putting (a) and (b) together show that f is a diffeomorphism of U onto $\mathbb{R}^n$.

5. Let $U \subseteq \mathbb{R}$ be the union of the open intervals, $k < x < k+1$, k an integer. Show that U *doesn't* have finite topology.

6. Let $V \subseteq \mathbb{R}^2$ be the open set obtained by deleting from $\mathbb{R}^2$ the points, $p_n = (0,n)$, n an integer. Show that V *doesn't* have finite topology. *Hint:* Let γ_n be a circle of radius $\frac{1}{2}$ centered about the point p_n. Using exercises 16–17 of §2.1 show that there exists a closed $\mathcal{C}^\infty$-one-form, ω_n on V with the property that $\int_{\gamma_n} \omega_n = 1$ and $\int_{\gamma_m} \omega_n = 0$ for $m \neq n$.

7. Let X be an n-dimensional manifold and $\mathbb{U} = \{U_i\,,\ i = 1,2\}$ a good cover of X. What are the cohomology groups of X if the nerve of this cover is

(a) $\{1\}$, $\{2\}$

(b) $\{1\}$, $\{2\}$, $\{1,2\}$?

8. Let X be an n-dimensional manifold and $\mathbb{U} = \{U_i\,,\ i = 1,2,3,\}$ a good cover of X. What are the cohomology groups of X if the nerve of this cover is

(a) $\{1\}$, $\{2\}$, $\{3\}$

(b) $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$

(c) $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$

(d) $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$

(e) $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$?

9. Let S^1 be the unit circle in $\mathbb{R}^3$ parametrized by arc length: $(x,y) = (\cos\theta, \sin\theta)$. Let U_1 be the set: $0 < \theta < \frac{2\pi}{3}$, U_2 the set: $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, and U_3 the set: $-\frac{2\pi}{3} < \theta < \frac{\pi}{3}$.

(a) Show that the U_i's are a good cover of S^1.

(b) Using the previous exercise compute the cohomology groups of S^1.

10. Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. Show that the sets

$$U_i = \{(x_1, x_2, x_3) \in S^2\,,\ x_i > 0\}$$

$i = 1, 2, 3$ and

$$U_i = \{(x_1, x_2, x_3) \in S^2\,,\ x_{i-3} < 0\}\,,$$

$i = 4, 5, 6$, are a good cover of S^2. What is the nerve of this cover?

11. Let X and Y be manifolds. Show that if they both have finite topology, their product, $X \times Y$, does as well.

12. (a) Let X be a manifold and let U_i, $i = 1, \ldots, N$, be a good cover of X. Show that $U_i \times \mathbb{R}$, $i = 1, \ldots, N$, is a good cover of $X \times \mathbb{R}$ and that the nerves of these two covers are the same.

(b) By Remark 5.3.13,

$$H^k(X \times \mathbb{R}) = H^k(X)\,.$$

Verify this directly using homotopy techniques.

(c) More generally, show that for all $\ell > 0$

$$H^k(X \times \mathbb{R}^\ell) = H^k(X) \tag{5.3.6}$$

(i) by concluding that this has to be the case in view of the Remark 5.3.13 and

(ii) by proving this directly using homotopy techniques.

5.4 Poincaré duality

In this chapter we've been studying two kinds of cohomology groups: the ordinary DeRham cohomology groups, H^k, and the compactly supported DeRham cohomology groups, H^k_c. It turns out that these groups are closely related. In fact if X is a connected, oriented n-dimensional manifold and has finite topology, $H^{n-k}_c(X)$ is the vector space dual of $H^k(X)$. We'll give a proof of this later in this section, however, before we do we'll need to review some basic linear algebra. Given two finite dimensional vector space, V and W, a *bilinear pairing* between V and W is a map

$$B : V \times W \to \mathbb{R} \tag{5.4.1}$$

which is linear in each of its factors. In other words, for fixed $w \in W$, the map

$$\ell_w : V \to \mathbb{R}\,, \quad \mathrm{v} \to B(\mathrm{v}, w) \tag{5.4.2}$$

is linear, and for $\mathrm{v} \in V$, the map

$$\ell_{\mathrm{v}} : W \to \mathbb{R}\,, \quad w \to B(\mathrm{v}, w) \tag{5.4.3}$$

is linear. Therefore, from the pairing (5.4.1) one gets a map

$$L_B : W \to V^*\,, \quad w \to \ell_w \tag{5.4.4}$$

and since $\ell_{w_1} + \ell_{w_2}(\mathrm{v}) = B(\mathrm{v}, w_1 + w_2) = \ell_{w_1+w_2}(\mathrm{v})$, this map is linear. We'll say that (5.4.1) is a non-singular pairing if (5.4.4) is bijective. Notice, by the way, that the roles of V and W can be reversed in this definition. Letting $B^\sharp(w, \mathrm{v}) = B(\mathrm{v}, w)$ we get an analogous linear map

$$L_{B^\sharp} : V \to W^* \tag{5.4.5}$$

and in fact

$$(L_{B^\sharp}(\mathrm{v}))(w) = (L_B(w))(\mathrm{v}) = B(\mathrm{v}, w)\,. \tag{5.4.6}$$

Thus if

$$\mu : V \to (V^*)^* \tag{5.4.7}$$

is the canonical identification of V with $(V^*)^*$ given by the recipe

$$\mu(\mathrm{v})(\ell) = \ell(\mathrm{v})$$

for $\mathrm{v} \in V$ and $\ell \in V^*$, we can rewrite (5.4.6) more suggestively in the form

$$L_{B^\sharp} = (L_B)^* \mu \tag{5.4.8}$$

i.e., L_B and $L_{B^\sharp}$ are just the transposes of each other. In particular L_B is bijective if and only if $L_{B^\sharp}$ is bijective.

Let's now apply these remarks to DeRham theory. Let X be a connected, oriented n-dimensional manifold. If X has finite topology the vector spaces, $H_c^{n-k}(X)$ and $H^k(X)$ are both finite dimensional. We will show that there is a natural bilinear pairing between these spaces, and hence by the discussion above, a natural linear mapping of $H^k(X)$ into the vector space dual of $H_c^{n-1}(X)$. To see this let c_1 be a cohomology class in $H_c^{n-k}(X)$ and c_2 a cohomology class in $H^k(X)$. Then by (1.1.43) their product, $c_1 \cdot c_2$, is an element of $H_c^n(X)$, and so by (1.1.8) we can define a pairing between c_1 and c_2 by setting

$$B(c_1, c_2) = I_X(c_1 \cdot c_2)\,. \tag{5.4.9}$$

Notice that if $\omega_1 \in \Omega_c^{n-k}(X)$ and $\omega_2 \in \Omega^k(X)$ are closed forms representing the cohomology classes, c_1 and c_2, then by (1.1.43) this pairing is given by the integral

$$B(c_1, c_2) = \int_X \omega_1 \wedge \omega_2\,. \tag{5.4.10}$$

We'll next show that this bilinear pairing is non-singular in one important special case:

Proposition 5.4.1. *If X is diffeomorphic to $\mathbb{R}^n$ the pairing defined by (5.4.9) is non-singular.*

Proof. To verify this there is very little to check. The vector spaces, $H^k(\mathbb{R}^n)$ and $H_c^{n-k}(\mathbb{R}^n)$ are zero except for $k = 0$, so all we have to check is that the pairing

$$H_c^n(X) \times H^0(X) \to \mathbb{R}$$

is non-singular. To see this recall that every compactly supported n-form is closed and that the only closed zero-forms are the constant functions, so at the level of forms, the pairing (5.4.9) is just the pairing

$$(\omega, c) \in \Omega^n(X) \times \mathbb{R} \to c \int_X \omega\,,$$

and this is zero if and only if c is zero or ω is in $d\Omega_c^{n-1}(X)$. Thus at the level of cohomology this pairing is non-singular.

□

We will now show how to prove this result in general.

Theorem 5.4.2 (Poincaré duality.)**.** *Let X be an oriented, connected n-dimensional manifold having finite topology. Then the pairing (5.4.9) is non-singular.*

The proof of this will be very similar in spirit to the proof that we gave in the last section to show that if X has finite topology its DeRham cohomology groups are finite dimensional. Like that proof, it involves Mayer–Victoris plus some elementary diagram-chasing. The "diagram-chasing" part of the proof consists of the following two lemmas.

Lemma 5.4.3. *Let V_1, V_2 and V_3 be finite dimensional vector spaces, and let $V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3$ be an exact sequence of linear mappings. Then the sequence of transpose maps*

$$V_3^* \xrightarrow{\beta^*} V_2^* \xrightarrow{\alpha^*} V_1$$

is exact.

Proof. Given a vector subspace, W_2, of V_2, let

$$W_2^\perp = \{\ell \in V_2^* \,;\, \ell(w) = 0 \text{ for } w \in W\}\,.$$

We'll leave for you to check that if W_2 is the kernel of β, then $W_2^\perp$ is the image of β^* and that if W_2 is the image of α, $W_2^\perp$ is the kernel of α^*. Hence if $\operatorname{Ker}\beta = \operatorname{Image}\alpha$, $\operatorname{Image}\beta^* = \operatorname{kernel}\alpha^*$.

□

Lemma 5.4.4 (the five lemma)**.** *Let the diagram below be a commutative diagram with the properties:*

(i) *All the vector spaces are finite dimensional.*

(ii) *The two rows are exact.*

(iii) *The linear maps, γ_i, $i = 1, 2, 4, 5$ are bijections.*

Then the map, γ_3, is a bijection.

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\uparrow{\scriptstyle\gamma_1} & & \uparrow{\scriptstyle\gamma_2} & & \uparrow{\scriptstyle\gamma_3} & & \uparrow{\scriptstyle\gamma_4} & & \uparrow{\scriptstyle\gamma_5} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5\,.
\end{array}$$

Proof. We'll show that γ_3 is surjective. Given $a_3 \in A_3$ there exists a $b_4 \in B_4$ such that $\gamma_4(b_4) = \alpha_3(a_3)$ since γ_4 is bijective. Moreover, $\gamma_5(\beta_4(b_4)) = \alpha_4(\alpha_3(a_3)) = 0$, by the exactness of the top row.

Therefore, since γ_5 is bijective, $\beta_4(b_4) = 0$, so by the exactness of the bottom row $b_4 = \beta_3(b_3)$ for some $b_3 \in B_3$, and hence

$$\alpha_3(\gamma_3(b_3)) = \gamma_4(\beta_3(b_3)) = \gamma_4(b_4) = \alpha_3(a_3)\,.$$

Thus $\alpha_3(a_3 - \gamma_3(b_3)) = 0$, so by the exactness of the top row

$$a_3 - \gamma_3(b_3) = \alpha_2(a_2)$$

for some $a_2 \in A_2$. Hence by the bijectivity of γ_2 there exists a $b_2 \in B_2$ with $a_2 = \gamma_2(b_2)$, and hence

$$a_3 - \gamma_3(b_3) = \alpha_2(a_2) = \alpha_2(\gamma_2(b_2)) = \gamma_3(\beta_2(b_2))\,.$$

Thus finally

$$a_3 = \gamma_3(b_3 + \beta_2(b_2))\,.$$

Since a_3 was any element of A_3 this proves the surjectivity of γ_3.

One can prove the injectivity of γ_3 by a similar diagram-chasing argument, but one can also prove this with less duplication of effort by taking the transposes of all the arrows in Figure 5.4.1 and noting that the same argument as above proves the surjectivity of $\gamma_3^* : A_3^* \to B_3^*$.

$\square$

To prove Theorem 5.4.2 we apply these lemmas to the diagram below. In this diagram U_1 and U_2 are open subsets of X, M is $U_1 \cup U_2$ and the vertical arrows are the mappings defined by the pairing (5.4.9). We will leave for you to check that this is a commutative diagram "up to sign". (To make it commutative one has to replace some of the vertical arrows, γ, by their negatives: $-\gamma$.) This is easy to check except for the commutative square on the extreme left. To check that this square commutes, some serious diagram-chasing is required.

$$\begin{tikzcd}
{} \arrow[r] & H^{n-(k-1)}(M) \arrow[r] & H^{n-k}(U_1 \cap U_2)^* \arrow[r] & H^{n-k}(U_1)^* \oplus H^{n-k}(U_2)^* \arrow[r] & H^{n-k}(M)^* \arrow[r] & {} \\
{} \arrow[r] & H_c^{k-1}(M) \arrow[r] \arrow[u] & H_c^k(U_1 \cap U_2) \arrow[r] \arrow[u] & H_c^k(U_1) \oplus H_c^k(U_2) \arrow[r] \arrow[u] & H_c^k(M) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

Figure 5.4.2

By Mayer–Victoris the bottom row of this figure is exact and by Mayer–Victoris and Lemma 5.4.3 the top row of this figure is exact. hence we can apply the "five lemma" to Figure 5.4.2 and conclude:

Lemma 5.4.5. *If the maps*

$$H^k(U) \to H_c^{n-k}(U)^* \tag{5.4.11}$$

defined by the pairing (5.4.9) are bijective for U_1, U_2 and $U_1 \cap U_2$, they are also bijective for $M = U_1 \cup U_2$.

Thus to prove Theorem 5.4.2 we can argue by induction as in § 5.3. Let $U_1, U_2, \dots, U_N$ be a good cover of X. If $N = 1$, then $X = U_1$ and, hence, since U_1 is diffeomorphic to $\mathbb{R}^n$, the map (5.4.12) is bijective by Proposition 5.4.1. Now let's assume the theorem is true for manifolds involving good covers by k open sets where k is less than N. Let $U' = U_1 \cup \cdots \cup U_{N-1}$ and $U'' = U_N$. Since

$$U' \cap U'' = U_1 \cap U_N \cup \cdots \cup U_{N-1} \cap U_N$$

it can be covered by a good cover by k open sets, $k < N$, and hence the hypotheses of the lemma are true for U', U'' and $U' \cap U''$. Thus the lemma says that (5.4.12) is bijective for the union, X, of U' and U''. □

Exercises.

1. (The "push-forward" operation in DeRham cohomology.) Let X be an m-dimensional manifold, Y an n-dimensional manifold and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Suppose that both of these manifolds are oriented and connected and have finite topology. Show that there exists a unique linear map

$$f_\sharp : H_c^{m-k}(X) \to H_c^{n-k}(Y) \tag{5.4.12}$$

with the property

$$B_Y(f_\sharp c_1, c_2) = B_X(c_1, f^\sharp c_2) \tag{5.4.13}$$

for all $c_1 \in H_c^{m-k}(X)$ and $c_2 \in H^k(Y)$. (In this formula B_X is the bilinear pairing (5.4.9) on X and B_Y is the bilinear pairing (5.4.9) on Y.)

2. Suppose that the map, f, in exercise 1 is proper. Show that there exists a unique linear map

$$f_\sharp : H^{m-k}(X) \to H^{n-k}(Y) \tag{5.4.14}$$

with the property

$$B_Y(c_1, f_\sharp c_2) = (-1)^{k(m-n)} B_X(f^\sharp c_1, c_2) \tag{5.4.15}$$

for all $c_1 \in H^k_c(Y)$ and $c_2 \in H^{m-k}(X)$, and show that, if X and Y are compact, this mapping is the same as the mapping, $f_\sharp$, in exercise 1.

3. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \times \mathbb{R} \to U$ be the projection, $f(x,t) = x$. Show that there is a unique linear mapping

$$f_* : \Omega^{k+1}_c(U \times \mathbb{R}) \to \Omega^k_c(U) \tag{5.4.16}$$

with the property

$$\int_U f_*\mu \wedge \nu = \int_{U\times\mathbb{R}} \mu \wedge f^*\nu \tag{5.4.17}$$

for all $\mu \in \Omega^{k+1}_c(U \times \mathbb{R})$ and $\nu \in \Omega^{n-k}(U)$.

Hint: Let $x_1, \dots, x_n$ and t be the standard coordinate functions on $\mathbb{R}^n \times \mathbb{R}$. By §2.2, exercise 5 every $(k+1)$-form, $\omega \in \Omega^{k+1}_c(U \times \mathbb{R})$ can be written uniquely in "reduced form" as a sum

$$\omega = \sum f_I \, dt \wedge dx_I + \sum g_J \, dx_J$$

over multi-indices, I and J, which are strictly increasing. Let

$$f_*\omega = \sum_I \left(\int_{\mathbb{R}} f_I(x,t) \, dt \right) dx_I \,. \tag{5.4.18}$$

4. Show that the mapping, f_*, in exercise 3 satisfies $f_* \, d\omega = df_*\omega$.

5. Show that if ω is a closed compactly supported $k+1$-form on $U \times \mathbb{R}$ then

$$[f_*\omega] = f_\sharp[\omega] \tag{5.4.19}$$

where $f_\sharp$ is the mapping (5.4.13) and f_* the mapping (5.4.17).

6. (a) Let U be an open subset of $\mathbb{R}^n$ and let $f : U \times \mathbb{R}^\ell \to U$ be the projection, $f(x,t) = x$. Show that there is a unique linear mapping

$$f_* : \Omega_c^{k+\ell}(U \times \mathbb{R}^\ell) \to \Omega_c^k(U) \tag{5.4.20}$$

with the property

$$\int_U f_*\mu \wedge \nu = \int_{U \times \mathbb{R}^\ell} \mu \wedge f^*\nu \tag{5.4.21}$$

for all $\mu \in \Omega_c^{k+\ell}(U \times \mathbb{R}^\ell)$ and $\nu \in \Omega^{n-k}(U)$.

Hint: Exercise 3 plus induction on ℓ.

(b) Show that for $\omega \in \Omega_c^{k+\ell}(U \times \mathbb{R}^\ell)$

$$df_*\omega = f_*\, d\omega\,.$$

(c) Show that if ω is a closed, compactly supported $k+\ell$-form on $X \times \mathbb{R}^\ell$

$$f_\sharp[\omega] = [f_*\omega] \tag{5.4.22}$$

where $f_\sharp : H_c^{k+\ell}(U \times \mathbb{R}^\ell) \to H_c^k(U)$ is the map (5.4.13).

7. Let X be an n-dimensional manifold and Y an m-dimensional manifold. Assume X and Y are compact, oriented and connected, and orient $X \times Y$ by giving it its natural product orientation. Let

$$f : X \times Y \to Y$$

be the projection map, $f(x,y) = y$. Given

$$\omega \in \Omega^m(X \times Y)$$

and $p \in Y$, let

$$f_*\omega(p) = \int_X \iota_p^*\omega \tag{5.4.23}$$

where $\iota_p : X \to X \times Y$ is the inclusion map, $\iota_p(x) = (x,p)$.

(a) Show that the function $f_*\omega$ defined by (5.5.24) is $\mathcal{C}^\infty$, i.e., is in $\Omega^0(Y)$.

(b) Show that if ω is closed this function is constant.

(c) Show that if ω is closed

$$[f_*\omega] = f_\sharp[\omega]$$

where $f_\sharp : H^n(X \times Y) \to H^0(Y)$ is the map (5.4.13).

8. (a) Let X be an n-dimensional manifold which is compact, connected and oriented. Combining Poincaré duality with exercise 12 in § 5.3 show that

$$H_c^{k+\ell}(X \times \mathbb{R}^\ell) = H_c^k(X)\,.$$

(b) Show, moreover, that if $f : X \times \mathbb{R}^\ell \to X$ is the projection, $f(x,a) = x$, then

$$f_\sharp : H_c^{k+\ell}(X \times \mathbb{R}^\ell) \to H_c^k(X)$$

is a bijection.

9. Let X and Y be as in exercise 1. Show that the push-forward operation (5.4.13) satisfies

$$f_\sharp; (c_1 \cdot f^\sharp c_2) = f_\sharp c_1 \cdot c_2$$

for $c_1 \in H_c^k(X)$ and $c_2 \in H^\ell(Y)$.

5.5 Thom classes and intersection theory

Let X be a connected, oriented n-dimensional manifold. If X has finite topology its cohomology groups are finite dimensional, and since the bilinear pairing, B, defined by (5.4.9) is non-singular we get from this pairing bijective linear maps

$$L_B : H_c^{n-k}(X) \to H^k(X)^* \tag{5.5.1}$$

and

$$L_B^* : H^{n-k}(X) \to H_c^k(X)^*\,. \tag{5.5.2}$$

In particular, if $\ell : H^k(X) \to \mathbb{R}$ is a linear function (i.e., an element of $H^k(X)^*$), then by (5.5.1) we can convert ℓ into a cohomology class

$$L_B^{-1}(\ell) \in H_c^{n-k}(X)\,, \tag{5.5.3}$$

and similarly if $\ell_c : H^k_c(X) \to \mathbb{R}$ is a linear function, we can convert it by (5.5.2) into a cohomology class

$$(L_B^*)^{-1}(\ell) \in H^{n-k}(X)\,. \tag{5.5.4}$$

One way that linear functions like this arise in practice is by integrating forms over submanifolds of X. Namely let Y be a closed, oriented k dimensional submanifold of X. Since Y is oriented, we have by (1.1.8) an integration operation in cohomology

$$I_Y : H^k_c(Y) \to \mathbb{R}\,,$$

and since Y is closed the inclusion map, ι_Y, of Y into X is proper, so we get from it a pull-back operation on cohomology

$$(\iota_Y)^\sharp : H^k_c(X) \to H^k_c(Y)$$

and by composing these two maps, we get a linear map, $\ell_Y = I_Y \circ (\iota_Y)^\sharp$, of $H^k_c(X)$ into $\mathbb{R}$. The cohomology class

$$T_Y = L_B^{-1}(\ell_Y) \in H^k_c(X) \tag{5.5.5}$$

associated with ℓ_Y is called the *Thom class* of the manifold, Y and has the defining property

$$B(T_Y, c) = I_Y(\iota_Y^\sharp c) \tag{5.5.6}$$

for $c \in H^k_c(X)$. Let's see what this defining property looks like at the level of forms. Let $\tau_Y \in \Omega^{n-k}(X)$ be a closed k-form representing T_Y. Then by (5.4.9), the formula (5.5.6), for $c = [\omega]$, becomes the integral formula

$$\int_X \tau_Y \wedge \omega = \int_Y \iota_Y^* \omega\,. \tag{5.5.7}$$

In other words, for every closed form, $\omega \in \Omega^{n-k}_c(X)$ the integral of ω over Y is equal to the integral over X of $\tau_Y \wedge \omega$. A closed form, τ_Y, with this "reproducing" property is called a *Thom form* for Y. Note that if we add to τ_Y an exact $(n-k)$-form, $\mu \in d\Omega^{n-k-1}(X)$, we get another representative, $\tau_Y + \mu$, of the cohomology class, T_Y, and hence another form with this reproducing property. Also, since the formula (5.5.7) is a direct translation into form language of the formula (5.5.6) any closed $(n-k)$-form, τ_Y, with the reproducing property (5.5.7) is a representative of the cohomology class, T_Y.

These remarks make sense as well for compactly supported cohomology. Suppose Y is compact. Then from the inclusion map we get a pull-back map

$$(\iota_Y)^\sharp : H^k(X) \to H^k(Y)$$

and since Y is compact, the integration operation, I_Y, is a map of $H^k(Y)$ into $\mathbb{R}$, so the composition of these two operations is a map,

$$\ell_Y : H^k(X) \to \mathbb{R}$$

which by (5.5.3) gets converted into a cohomology class

$$T_Y = L_B^{-1}(\ell_Y) \in H_c^{n-k}(X)\,.$$

Moreover, if $\tau_Y \in \Omega_c^{n-k}(X)$ is a closed form, it represents this cohomology class if and only if it has the reproducing property

$$\int_X \tau_Y \wedge \omega = \int_Y \iota_Y^* \omega \tag{5.5.8}$$

for closed forms, ω, in $\Omega^{n-k}(X)$. (There's a subtle difference, however, between formula (5.5.7) and formula (5.5.8). In (5.5.7) ω has to be closed *and* compactly supported and in (5.5.8) it just has to be closed.)

As above we have a lot of latitude in our choice of τ_Y: we can add to it any element of $d\Omega_c^{n-k-1}(X)$. One consequence of this is the following.

Theorem 5.5.1. *Given a neighborhood, U, of Y in X there exists a closed form, $\tau_Y \in \Omega_c^{n-k}(U)$, with the reproducing property*

$$\int_U \tau_Y \wedge \omega = \int_Y \iota_Y^* \omega \tag{5.5.9}$$

for closed forms, $\omega \in \Omega^k(U)$.

Hence in particular, τ_Y has the reproducing property (5.5.8) for closed forms, $\omega \in \Omega^{n-k}(X)$. This result shows that the Thom form, τ_Y, can be chosen to have support in an *arbitrarily small neighborhood* of Y. To prove Theorem 5.5.1 we note that by Theorem 5.3.8 we can assume that U has finite topology and hence, in our definition of τ_Y, we can replace the manifold, X, by the open submanifold,

U. This gives us a Thom form, τ_Y, with support in U and with the reproducing property (5.5.9) for closed forms $\omega \in \Omega^{n-k}(U)$.

□

Let's see what Thom forms actually look like in concrete examples. Suppose Y is defined globally by a system of ℓ independent equations, i.e., suppose there exists an open neighborhood, $\mathcal{O}$, of Y in X, a $\mathcal{C}^\infty$ map, $f : \mathcal{O} \to \mathbb{R}^\ell$, and a bounded open convex neighborhood, V, of the origin in $\mathbb{R}^n$ such that

$$\text{(5.5.10)} \qquad \begin{aligned} &\text{(i)} \quad \text{The origin is a regular value of } f\,. \\ &\text{(ii)} \quad f^{-1}(\bar{V}) \text{ is closed in } X\,. \\ &\text{(iii)} \quad Y = f^{-1}(0)\,. \end{aligned}$$

Then by (i) and (iii) Y is a closed submanifold of $\mathcal{O}$ and by (ii) it's a closed submanifold of X. Moreover, it has a natural orientation: For every $p \in Y$ the map

$$df_p : T_pX \to T_0\mathbb{R}^\ell$$

is surjective, and its kernel is T_pY, so from the standard orientation of $T_0\mathbb{R}^\ell$ one gets an orientation of the quotient space,

$$T_pX/T_pY\,,$$

and hence since T_pX is oriented, one gets, by Theorem 1.9.4, an orientation on T_pY. (See §4.4, example 2.) Now let μ be an element of $\Omega_c^\ell(X)$. Then $f^*\mu$ is supported in $f^{-1}(\bar{V})$ and hence by property (ii) of (5.5.10) we can extend it to X by setting it equal to zero outside $\mathcal{O}$. We will prove

Theorem 5.5.2. *If*

$$\text{(5.5.11)} \qquad \int_V \mu = 1\,,$$

$f^*\mu$ is a Thom form for Y.

To prove this we'll first prove that if $f^*\mu$ has property (5.5.7) for some choice of μ it has this property for every choice of μ.

Lemma 5.5.3. *Let μ_1 and μ_2 be forms in $\Omega_c^\ell(V)$ with the property (5.5.11). Then for every closed k-form, $\nu \in \Omega_c^k(X)$*

$$\int_X f^*\mu_1 \wedge \nu = \int_X f^*\mu_2 \wedge \nu\,.$$

Proof. By Theorem 3.2.1, $\mu_1 - \mu_2 = d\beta$ for some $\beta \in \Omega_c^{\ell-1}(V)$, hence, since $d\nu = 0$

$$(f^*\mu_1 - f^*\mu_2) \wedge \nu = df^*\beta \wedge \nu = d(f^*\beta \wedge \nu)\,.$$

Therefore, by Stokes theorem, the integral over X of the expression on the left is zero.

□

Now suppose $\mu = \rho(x_1, \ldots, x_\ell)\, dx_1 \wedge \cdots \wedge dx_\ell$, for ρ in $\mathcal{C}_0^\infty(V)$. For $t \leq 1$ let

$$\mu_t = t^\ell \rho\left(\frac{x_1}{t}, \cdots, \frac{x_\ell}{t}\right) dx_1 \wedge \cdots dx_\ell\,. \tag{5.5.12}$$

This form is supported in the convex set, tV, so by Lemma 5.5.3

$$\int_X f^*\mu_t \wedge \nu = \int_X f^*\mu \wedge \nu \tag{5.5.13}$$

for all closed forms $\nu \in \Omega_c^k(X)$. Hence to prove that $f^*\mu$ has the property (5.5.7) it suffices to prove

$$\mathrm{Lim}_{t \to 0} \int f^*\mu_t \wedge \nu = \int_Y \iota_Y^*\nu\,. \tag{5.5.14}$$

We'll prove this by proving a stronger result.

Lemma 5.5.4. *The assertion (5.5.14) is true for every k-form $\nu \in \Omega_c^k(X)$.*

Proof. The canonical form theorem for submersions (see Theorem 4.3.6) says that for every $p \in Y$ there exists a neighborhood U_p of p in Y, a neighborhood, W of 0 in $\mathbb{R}^n$, and an orientation preserving diffeomorphism $\psi : (W, 0) \to (U_p, p)$ such that

$$f \circ \psi = \pi \tag{5.5.15}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^\ell$ is the canonical submersion, $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_\ell)$. Let $\mathbb{U}$ be the cover of $\mathcal{O}$ by the open sets, $\mathcal{O} - Y$ and the U_p's. Choosing a partition of unity subordinate to this cover it suffices to verify (5.5.14) for ν in $\Omega_c^k(\mathcal{O} - Y)$ and ν in $\Omega_c^k(U_p)$. Let's first suppose ν is in $\Omega_c^k(\mathcal{O} - Y)$. Then $f(\operatorname{supp} \nu)$ is a compact subset of $\mathbb{R}^\ell - \{0\}$ and hence for t small $f(\operatorname{supp} \nu)$ is disjoint from tV, and both sides of (5.5.14) are zero. Next suppose that ν is in $\Omega_c^k(U_p)$. Then $\psi^*\nu$ is a compactly supported k-form on W so we can write it as a sum

$$\psi^*\nu = \sum h_I(x)\, dx_I\,, \quad h_I \in \mathcal{C}_0^\infty(W)$$

the I's being strictly increasing multi-indices of length k. Let $I_0 = (\ell+1, \ell_2+2, \dots, n)$. Then

$$\pi^*\mu_t \wedge \psi^*\nu = t^\ell \rho(\frac{x_1}{t}\,, \cdots\,, \frac{x_\ell}{t}) h_{I_0}(x_1, \dots, x_n)\, dx_r \wedge \cdots dx_n \tag{5.5.16}$$

and by (5.5.15)

$$\psi^*(f^*\mu_t \wedge \nu) = \pi^*\mu_t \wedge \psi^*\nu$$

and hence since ψ is orientation preserving

$$\begin{aligned}\int_{U_p} f^*\mu_t \wedge \nu &= t^\ell \int_{\mathbb{R}^n} \rho\left(\frac{x_1}{t}\,, \cdots\,, \frac{x_\ell}{t}\right) h_{I_0}(x_1, \dots, x_n)\, dx \\ &= \int_{\mathbb{R}^n} \rho(x_1, \dots, x_\ell) h_{I_0}(tx_1, \dots, tx_\ell\,,\, x_{\ell+1}, \dots, x_n)\, dx\end{aligned}$$

and the limit of this expression as t tends to zero is

$$\int \rho(x_1, \dots, x_\ell) h_{I_0}(0, \dots, 0\,,\, x_{\ell+1}, \dots, x_n)\, dx_1 \dots\, dx_n$$

or

$$\int h_I(0, \dots, 0\,,\, x_{\ell+1}, \dots, x_n)\, dx_{\ell+1} \cdots\, dx_n\,. \tag{5.5.17}$$

This, however, is just the integral of $\psi^*\nu$ over the set $\pi^{-1}(0) \cap W$. By (5.5.14) ψ maps this set diffeomorphically onto $Y \cap U_p$ and by our recipe for orienting Y this diffeomorphism is an orientation-preserving diffeomorphism, so the integral (5.5.17) is equal to the integral of ν over Y.

□

We'll now describe some applications of Thom forms to *topological intersection theory*. Let Y and Z be closed, oriented submanifolds of X of dimensions k and ℓ where $k + \ell = n$, and let's assume one of them (say Z) is compact. We will show below how to define an "intersection number", $I(Y, Z)$, which on the one hand will be a topological invariant of Y and Z and on the other hand will actually count, with appropriate $\pm$-signs, the number of points of intersection of Y and Z when they intersect non-tangentially. (Thus this notion is similar to the notion of "degree f" for a $\mathcal{C}^\infty$ mapping f. On the one hand "degree f" is a topological invariant of f. It's unchanged if we deform f by a homotopy. On the other hand if q is a regular value of f, "degree f" counts with appropriate $\pm$-signs the number of points in the set, $f^{-1}(q)$.)

We'll first give the topological definition of this intersection number. This is by the formula

$$I(Y,Z) = B(T_Y, T_Z) \tag{5.5.18}$$

where $T_Y \in H^\ell(X)$ and $T_Z \in H^k_c(X)$ and B is the bilinear pairing (5.4.9). If $\tau_Y \in \Omega^\ell(X)$ and $\tau_Z \in \Omega^k_c(X)$ are Thom forms representing T_Y and T_Z, (5.5.18) can also be defined as the integral

$$I(Y,Z) = \int_X \tau_Y \wedge \tau_Z \tag{5.5.19}$$

or by (5.5.9), as the integral over Y,

$$I(Y,Z) = \int_Y \iota_Y^* \tau_Z \tag{5.5.20}$$

or, since $\tau_Y \wedge \tau_Z = (-1)^{k\ell} \tau_Z \wedge \tau_Y$, as the integral over Z

$$I(X,Y) = (-1)^{k\ell} \int_Z \iota_Z^* \tau_Y \,. \tag{5.5.21}$$

In particular

$$I(Y,Z) = (-1)^{k\ell} I(Z,Y). \tag{5.5.22}$$

As a test case for our declaring $I(Y,Z)$ to be the intersection number of Y and Z we will first prove:

Proposition 5.5.5. *If Y and Z* don't *intersect, then $I(Y,Z) = 0$.*

Proof. If Y and Z don't intersect then, since Y is closed, $U = X - Y$ is an open neighborhood of Z in X, therefore since Z is compact there exists by Theorem 5.5.1 a Thom form, τ_Z in $\Omega^\ell_c(U)$. Thus $\iota_Y^* \tau_Z = 0$, and so by (5.5.20) $I(Y,Z) = 0$.

$\square$

We'll next indicate how one computes $I(Y,Z)$ when Y and Z intersect "non-tangentially", or, to use terminology more in current usage, when their intersection is *transversal*. Recall that at a point of intersection, $p \in Y \cap Z$, T_pY and T_pZ are vector subspaces of T_pX.

Definition 5.5.6. *Y and Z intersect* transversally *if for every $p \in Y \cap Z$, $T_pY \cap T_pZ = \{0\}$.*

Since $n = k + \ell = \dim T_pY + \dim T_pZ = \dim T_pX$, this condition is equivalent to

$$T_pX = T_pY \oplus T_pZ\,, \tag{5.5.23}$$

i.e., every vector, $u \in T_pX$, can be written uniquely as a sum, $u = \mathrm{v} + w$, with $\mathrm{v} \in T_pY$ and $w \in T_pZ$. Since X, Y and Z are oriented, their tangent spaces at p are oriented, and we'll say that these spaces are *compatibly oriented* if the orientations of the two sides of (5.5.23) agree. (In other words if $\mathrm{v}_1, \dots, \mathrm{v}_k$ is an oriented basis of T_pY and $w_1, \dots, w_\ell$ is an oriented basis of T_pZ, the n vectors, $\mathrm{v}_1, \dots, \mathrm{v}_k$, $w_1, \dots, w_\ell$, are an oriented basis of T_pX.) We will define the *local intersection number*, $I_p(Y,Z)$, of Y and Z at p to be equal to $+1$ if X, Y and Z are compatibly oriented at p and to be equal to -1 if they're not. With this notation we'll prove

Theorem 5.5.7. *If Y and Z intersect transversally then $Y \cap Z$ is a finite set and*

$$I(Y,Z) = \sum_{p \in Y \cap Z} I_p(Y,Z). \tag{5.5.24}$$

To prove this we first need to show that transverse intersections look nice locally.

Theorem 5.5.8. *If Y and Z intersect transversally, then for every $p \in Y \cap Z$, there exists an open neighborhood, V_p, of p in X, an open neighborhood, U_p, of the origin in $\mathbb{R}^n$ and an orientation preserving diffeomorphism*

$$\psi_p : V_p \to U_p$$

which maps $V_p \cap Y$ diffeomorphically onto the subset of U_p defined by the equations: $x_1 = \cdots = x_\ell = 0$, and maps $V \cap Z$ onto the subset of U_p defined by the equations: $x_{\ell+1} = \cdots = x_n = 0$.

Proof. Since this result is a local result, we can assume that X is $\mathbb{R}^n$ and hence by Theorem 4.2.7 that there exists a neighborhood, V_p, of p in $\mathbb{R}^n$ and submersions $f : (V_p, p) \to (\mathbb{R}^\ell, 0)$ and $g : (V_p, p) \to (\mathbb{R}^k, 0)$ with the properties

$$V_p \cap Y = f^{-1}(0) \tag{5.5.25}$$

and

$$\mathrm{v}_p \cap Z = g^{-1}(0)\,. \tag{5.5.26}$$

Moreover, by (4.3.4)

$$T_pY = (df_p)^{-1}(0)$$

and

$$T_pZ = (dg_p)^{-1}(0)\,.$$

Hence by (5.5.23), the equations

$$df_p(\mathrm{v}) = dg_p(\mathrm{v}) = 0 \tag{5.5.27}$$

for $\mathrm{v} \in T_pX$ imply that $\mathrm{v} = 0$. Now let $\psi_p : V_p \to \mathbb{R}^n$ be the map

$$(f, g) : V_p \to \mathbb{R}^\ell \times \mathbb{R}^k = \mathbb{R}^n\,.$$

Then by (5.5.27), $d\psi_p$ is bijective, therefore, shrinking V_p if necessary, we can assume that ψ_p maps V_p diffeomorphically onto a neighborhood, U_p, of the origin in $\mathbb{R}^n$, and hence by (5.5.25) and (5.5.26), ψ_p maps $V_p \cap Y$ onto the set: $x_1 = \cdots = x_\ell = 0$ and maps $V_p \cap Z$ onto the set: $x_{\ell+1} = \cdots = x_n = 0$. Finally, if ψ isn't orientation preserving, we can make it so by composing it with the involution, $(x_1, \ldots, x_n) \to (x_1, x_2, \ldots, x_{n-1}, -x_n)$.

□

From this result we deduce:

Theorem 5.5.9. *If Y and Z intersect transversally, their intersection is a finite set.*

Proof. By Theorem 5.5.8 the only point of intersection in V_p is p itself. Moreover, since Y is closed and Z is compact, $Y \cap Z$ is compact.

Therefore, since the V_p's cover $Y \cap Z$ we can extract a finite subcover by the Heine–Borel theorem. However, since no two V_p's cover the same point of $Y \cap Z$, this cover must already be a finite subcover.

□

We will now prove Theorem 5.5.7. Since Y is closed, the map, $\iota_Y : Y \to X$ is proper, so by Theorem 3.4.2 there exists a neighborhood, U, of Z in X such that $U \cap Y$ is contained in the union of the open sets, V_p, above. Moreover by Theorem 5.5.1 we can choose τ_Z to be supported in U and by Theorem 5.3.2 we can assume that U has finite topology, so we're reduced to proving the theorem with X replaced by U and Y replaced by $Y \cap U$. Let

$$\mathcal{O} = \left(\bigcup V_p\right) \cap U \,,$$

let

$$f : \mathcal{O} \to \mathbb{R}^\ell$$

be the map whose restriction to $V_p \cap U$ is $\pi \circ \psi_p$ where π is, as in (5.5.15), the canonical submersion of $\mathbb{R}^n$ onto $\mathbb{R}^\ell$, and finally let V be a bounded convex neighborhood of $\mathbb{R}^\ell$, whose closure is contained in the intersection of the open sets, $\pi \circ \psi_p(V_p \cap U)$. Then $f^{-1}(\bar{V})$ is a closed subset of U, so if we replace X by U and Y by $Y \cap U$, the data $(f, \mathcal{O}, V)$ satisfy the conditions (5.5.10). Thus to prove Theorem 5.5.7 it suffices by Theorem 5.5.2 to prove this theorem with

$$\tau_Y = \sigma_p(Y) f^* \mu$$

on $V_p \cap \mathcal{O}$ where $\sigma_p(Y) = +1$ or -1 depending on whether the orientation of $Y \cap V_p$ in Theorem 5.5.2 coincides with the given orientation of Y or not. Thus

$$\begin{aligned}
I(Y,Z) &= (-1)^{k\ell} I(Z,Y) \\
&= (-1)^{k\ell} \sum_p \sigma_p(Y) \int_Z \iota_Z^* f^* \mu \\
&= (-1)^{k\ell} \sum_p \sigma_p(Y) \int_Z \iota_Z^* \psi_p^* \pi^* \mu \\
&= \sum_p (-1)^{k\ell} \sigma_p(Y) \int_{Z \cap V_p} (\pi \circ \psi_p \circ \iota_Z)^* \mu \,.
\end{aligned}$$

But $\pi \circ \psi_p \circ \iota_Z$ maps an open neighborhood of p in $U_p \cap Z$ diffeomorphically onto V, and μ is compactly supported in V so by (5.5.11)

$$\int_{Z \cap U_p} (\pi \circ \psi_p \circ \iota_Z)^* \mu = \sigma_p(Z) \int_V \mu = \sigma_p(Z)$$

where $\sigma_p(Z) = +1$ or -1 depending on whether $\pi \circ \psi_p \circ \iota_Z$ is orientation preserving or not. Thus finally

$$I(Y, Z) = \sum (-1)^{k\ell} \sigma_p(Y) \sigma_p(Z) \,.$$

We will leave as an exercise the task of unraveling these orientations and showing that

$$(-1)^{k\ell} \sigma_p(Y) \sigma_p(Z) = I_p(Y, Z)$$

and hence that $I(Y, Z) = \sum_p I_p(Y, Z)$.

Exercises.

1. Let X be a connected, oriented n-dimensional manifold, W a connected, oriented ℓ-dimensional manifold, $f : X \to W$ a $\mathcal{C}^\infty$ map, and Y a closed submanifold of X of dimension $k = n - \ell$. Suppose Y is a "level set" of the map, f, i.e., suppose that q is a regular value of f and that $Y = f^{-1}(q)$. Show that if μ is in $\Omega_c^\ell(Z)$ and its integral over Z is 1, then one can orient Y so that $\tau_Y = f^*\mu$ is a Thom form for Y.

   Hint: Theorem 5.5.2.

2. In exercise 1 show that if $Z \subseteq X$ is a compact oriented ℓ-dimensional submanifold of X then

$$I(Y, Z) = (-1)^{k\ell} \deg(f \circ \iota_Z) \,.$$

3. Let q_1 be another regular value of the map, $f : X \to W$, and let $Y_1 = f^{-1}(q)$. Show that

$$I(Y, Z) = I(Y_1, Z) \,.$$

4. (a) Show that if q is a regular value of the map, $f \circ \iota_Z : Z \to W$ then Z and Y intersect transversally.

(b) Show that this is an "if and only if" proposition: If Y and Z intersect transversally then q is a regular value of the map, $f \circ \iota_Z$.

5. Suppose q is a regular value of the map, $f \circ \iota_Z$. Show that p is in $Y \cap Z$ if and only if p is in the pre-image $(f \circ \iota_Z)^{-1}(q)$ of q and that

$$I_p(X, Y) = (-1)^{k\ell} \sigma_p$$

where σ_p is the orientation number of the map, $f \circ \iota_Z$, at p, i.e., $\sigma_p = 1$ if $f \circ \iota_Z$ is orientation-preserving at p and $\sigma_p = -1$ if $f \circ \iota_Z$ is orientation-reversving at p.

6. Suppose the map $f : X \to W$ is proper. Show that there exists a neighborhood, V, of q in W having the property that *all* points of V are regular values of f.

Hint: Since q is a regular value of f there exists, for every $p \in f^{-1}(q)$ a neighborhood, U_p of p, on which f is a submersion. Conclude, by Theorem 3.4.2, that there exists a neighborhood, V, of q with $f^{-1}(V) \subseteq \bigcup U_p$.

7. Show that in every neighborhood, V_1, of q in V there exists a point, q_1, whose pre-image

$$Y_1 = f^{-1}(q_1)$$

intersects Z transversally. (*Hint:* Exercise 4 plus Sard's theorem.) Conclude that one can "deform Y an arbitrarily small amount so that it intersects Z transversally".

8. (Intersection theory for mappings.) Let X be an oriented, connected n-dimensional manifold, Z a compact, oriented ℓ-dimensional submanifold, Y an oriented manifold of dimension $k = n - \ell$ and $f : Y \to X$ a proper $\mathcal{C}^\infty$ map. Define the intersection number of f with Z to be the integral

$$I(f, Z) = \int_Y f^* \tau_Z \,.$$

(a) Show that $I(f, Z)$ is a homotopy invariant of f, i.e., show that if $f_i : Y \to X$, $i = 0, 1$ are proper $\mathcal{C}^\infty$ maps and are properly homotopic, then

$$I(f_0, Z) = I(f_1, Z) \,.$$

(b) Show that if Y is a closed submanifold of X of dimension $k = n - \ell$ and $\iota_Y : Y \to X$ is the inclusion map

$$I(\iota_Y, Z) = I(Y, Z)\,.$$

9. (a) Let X be an oriented, connected n-dimensional manifold and let Z be a compact zero-dimensional submanifold consisting of a single point, $z_0 \in X$. Show that if μ is in $\Omega_c^n(X)$ then μ is a Thom form for Z if and only if its integral is 1.

(b) Let Y be an oriented n-dimensional manifold and $f : Y \to X$ a $\mathcal{C}^\infty$ map. Show that for $Z = \{z_0\}$ as in part a

$$I(f, Z) = \deg(f)\,.$$

5.6 The Lefshetz theorem

In this section we'll apply the intersection techniques that we developed in §5.5 to a concrete problem in dynamical systems: counting the number of fixed points of a differentiable mapping. The Brouwer fixed point theorem, which we discussed in §3.6, told us that a $\mathcal{C}^\infty$ map of the unit ball into itself has to have at least one fixed point. The Lefshetz theorem is a similar result for manifolds. It will tell us that a $\mathcal{C}^\infty$ map of a compact manifold into itself has to have a fixed point if a certain topological invariant of the map, its *global Lefshetz number*, is non-zero.

Before stating this result, we will first show how to translate the problem of counting fixed points of a mapping into an intersection number problem. Let X be an oriented, compact n-dimensional manifold and $f : X \to X$ a $\mathcal{C}^\infty$ map. Define the *graph of* f in $X \times X$ to be the set

$$\Gamma_f = \{(x, f(x))\,; \quad x \in X\}\,. \tag{5.6.1}$$

It's easy to see that this is an n-dimensional submanifold of $X \times X$ and that this manifold is diffeomorphic to X itself. In fact, in one direction, there is a $\mathcal{C}^\infty$ map

$$\gamma_f : X \to \Gamma_f\,, \quad \gamma_f(x) = (x, f(x))\,, \tag{5.6.2}$$

and, in the other direction, a $\mathcal{C}^\infty$ map

$$\pi : \Gamma_f \to X\,, \quad (x, f(x)) \to x\,, \tag{5.6.3}$$

and it's obvious that these maps are inverses of each other and hence diffeomorphisms. We will orient Γ_f by requiring that γ_f and π be orientation-preserving diffeomorphisms.

An example of a graph is the graph of the identity map of X onto itself. This is the *diagonal* in $X \times X$

$$\Delta = \{(x,x)\,,\ x \in X\} \tag{5.6.4}$$

and its intersection with Γ_f is the set

$$\{(x,x)\,,\ f(x) = x\}\,, \tag{5.6.5}$$

which is just the set of fixed points of f. Hence a natural way to count the fixed points of f is as the intersection number of Γ_f and Δ in $X \times X$. To do so we need these three manifolds to be oriented, but, as we noted above, Γ_f and Δ acquire orientations from the identifications (5.6.2) and, as for $X \times X$, we'll give it its natural orientation as a product of oriented manifolds. (See §4.5.)

Definition 5.6.1. *The global Lefshetz number of X is the intersection number*

$$L(f) = I(\Gamma_f, \Delta)\,. \tag{5.6.6}$$

In this section we'll give two recipes for computing this number: one by topological methods and the other by making transversality assumptions and computing this number as a sum of local intersection numbers a la (5.5.24). We'll first show what one gets from the transversality approach.

Definition 5.6.2. *The map, f, is a Lefshetz map if Γ_f and Δ intersect transversally.*

Let's see what being Lefshetz entails. Suppose p is a fixed point of f. Then at $q = (p,p) \in \Gamma_f$

$$T_q(\Gamma_f) = (d\gamma_f)_p T_p X = \{(\mathrm{v},\, df_p(\mathrm{v}))\,,\ \mathrm{v} \in T_p X\} \tag{5.6.7}$$

and, in particular, for the identity map,

$$T_q(\Delta) = \{(\mathrm{v},\mathrm{v})\,,\ \mathrm{v} \in T_p X\}\,. \tag{5.6.8}$$

Therefore, if Δ and Γ_f are to intersect transversally, the intersection of (5.6.7) and (5.6.8) inside $T_q(X \times X)$ has to be the zero space. In other words if

$$(\mathrm{v}, df_p(\mathrm{v})) = (\mathrm{v},\mathrm{v}) \tag{5.6.9}$$

then $\mathrm{v} = 0$. But the identity (5.6.9) says that v is a fixed point of df_p, so transversality at p amounts to the assertion

$$df_p(\mathrm{v}) = \mathrm{v} \Leftrightarrow \mathrm{v} = 0\,, \tag{5.6.10}$$

or in other words the assertion that the map

$$(I - df_p) : T_pX \to T_pX \tag{5.6.11}$$

is bijective. We'll now prove

Proposition 5.6.3. *The local intersection number $I_p(\Gamma_f, \Delta)$ is 1 if (5.6.11) is orientation-preserving and -1 if not.*

In other words $I_p(\Gamma_f, \Delta)$ is the sign of $\det(I - df_p)$. To prove this let $e_1, \ldots, e_n$ be an oriented basis of T_pX and let

$$df_p(e_i) = \sum a_{j,i} e_j\,. \tag{5.6.12}$$

Now set

$$\mathrm{v}_i = (e_i, 0) \in T_q(X \times X)$$

and

$$w_i = (0, e_i) \in T_q(X \times X)\,.$$

Then by the deifnition of the product orientation on $X \times X$

$$\mathrm{v}_1, \ldots, \mathrm{v}_n\,,\ w_1, \ldots, w_n \tag{5.6.13}$$

is an oriented basis of $T_q(X \times X)$ and by (5.6.7)

$$\mathrm{v}_1 + \sum a_{j,i} w_j \ldots, \mathrm{v}_n + \sum a_{j,n} w_j \tag{5.6.14}$$

is an oriented basis of $T_q\Gamma_f$ and

$$\mathrm{v}_1 + w_1, \ldots, \mathrm{v}_n + w_n \tag{5.6.15}$$

is an oriented basis of $T_q\Delta$. Thus $I_p(\Gamma_f, \Delta) = +1$ or -1 depending on whether or not the basis

$$\mathrm{v}_1 + \sum a_{j,i} w_j, \ldots, \mathrm{v}_n + \sum a_{j,n} w_j, \mathrm{v}_1 + w_1, \ldots, \mathrm{v}_n + w_n$$

of $T_q(X \times X)$ is compatibly oriented with the basis (5.6.12). Thus $I_p(\Gamma_f, \Delta) = +1$ or -1 depending on whether the determinant of the $2n \times 2n$ matrix relating these two bases:

$$\begin{bmatrix} I & A \\ I\,, & I \end{bmatrix}, \quad A = [a_{i.j}] \tag{5.6.16}$$

is positive or negative. However, it's easy to see that this determinant is equal to $\det(I - A)$ and hence by (5.6.12) to $\det(I - df_p)$. *Hint:* By elementary row operations (5.6.16) can be converted into the matrix

$$\begin{bmatrix} I\,, & A \\ 0\,, & I - A \end{bmatrix}.$$

□

Let's summarize what we've shown so far.

Theorem 5.6.4. *The map, $f : X \to X$, is a Lefshetz map if and only if, for every fixed point, p, the map*

$$I - df_p : T_pX \to T_pX \tag{$*$}$$

is bijective. Moreover for Lefshetz maps

$$L(f) = \sum_{p - f_{(p)}} L_p(f) \tag{5.6.17}$$

where $L_p(f) = +1$ if $(*)$ is orientation-preserving and -1 if it's orientation-reversing.

We'll next describe how to compute $L(f)$ as a topological invariant of f. Let ι_Γ be the inclusion map of Γ_f into $X \times X$ and let $T_\Delta \in H^n(X \times X)$ be the Thom class of Δ. Then by (5.5.20)

$$L(f) = I_{\Gamma_f}(\iota^* T_\Delta)$$

and hence since the mapping, $\gamma_f : X \to X \times X$ defined by (5.6.2) is an orientation-preserving diffeomorphism of X onto Γ_f

$$L(f) = I_X(\gamma_f^* T_\Delta)\,. \tag{5.6.18}$$

To evaluate the expression on the right we'll need to know some facts about the cohomology groups of product manifolds. The main result on this topic is the "Künneth" theorem, and we'll take up the formulation and proof of this theorem in §5.7. First, however, we'll describe a result which follows from the Künneth theorem and which will enable us to complete our computation of $L(f)$.

Let π_1 and π_2 be the projection of $X \times X$ onto its first and second factors, i.e., let

$$\pi_i : X \times X \to X \qquad i = 1, 2$$

be the map, $\pi_i(x_1, x_2) = x_i$. Then by (5.6.2)

$$\pi_1 \cdot \gamma_f = i\, d_X \tag{5.6.19}$$

and

$$\pi_2 \cdot \gamma_f = f\,. \tag{5.6.20}$$

Lemma 5.6.5. *If ω_1 and ω_2 are in $\Omega^n(X)$ then*

$$\int_{X \times X} \pi_1^* \omega_1 \wedge \pi_2^* \omega_2 = \left(\int_X \omega_1 \right) \left(\int_X \omega_2 \right)\,. \tag{5.6.21}$$

Proof. By a partition of unity argument we can assume that ω_i has compact support in a parametrizable open set, V_i. Let U_i be an open subset of $\mathbb{R}^n$ and $\varphi_i : U_i \to V_i$ an orientation-preserving diffeomorphism. Then

$$\varphi_i^* \omega = \rho_i\, dx_1 \wedge \cdots \wedge dx_n$$

with $\rho_i \in \mathcal{C}_0^\infty(U_i)$, so the right hand side of (5.6.21) is the product of integrals over $\mathbb{R}^n$:

$$\int \rho_1(x)\, dx \int \rho_2(x)\, dx\,. \tag{5.6.22}$$

Moreover, since $X \times X$ is oriented by its product orientation, the map

$$\psi : U_1 \times U_2 \to V_1 \times V_2$$

mapping (x, y) to $(\varphi_1(x)\,, \varphi_2(y))$ is an orientation-preserving diffeomorphism and since $\pi_i \circ \psi = \varphi_i$

$$\begin{aligned} \psi^*(\pi_1^* \omega_1 \wedge \pi_2^* \omega_2) &= \varphi_1^* \omega_1 \wedge \varphi_2^* \omega_2 \\ &= \rho_1(x) \rho_2(y)\, dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \end{aligned}$$

and hence the left hand side of (5.6.21) is the integral over $\mathbb{R}^{2n}$ of the function, $\rho_1(x)\rho_2(y)$, and therefore, by integration by parts, is equal to the product (5.6.22).

□

As a corollary of this lemma we get a product formula for cohomology classes:

Lemma 5.6.6. *If c_1 and c_2 are in $H^n(X)$ then*

$$I_{X\times X}(\pi_1^* c_1 \cdot \pi_2^* c_2) = I_X(c_1) I_X(c_2)\,. \tag{5.6.23}$$

Now let $d_k = \dim H^k(X)$ and note that since X is compact, Poincaré duality tells us that $d_k = d_\ell$ when $\ell = n-k$. In fact it tells us even more. Let

$$\mu_i^k\,, \quad i = 1, \ldots, d_k$$

be a basis of $H^k(X)$. Then, since the pairing (5.4.9) is non-singular, there exists for $\ell = n-k$ a "dual" basis

$$\nu_j^\ell\,, \quad j = 1, \ldots, d_\ell$$

of $H^\ell(X)$ satisfying

$$I_X(\mu_i^k \cdot \nu_j^\ell) = \delta_{ij}\,. \tag{5.6.24}$$

Lemma 5.6.7. *The cohomology classes*

$$\pi_1^\sharp \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k\,, \quad k + \ell = n \tag{5.6.25}$$

for $k = 0, \ldots, n$ and $1 \le r, s \le d_k$, are a basis for $H^n(X \times X)$.

This is the corollary of the Künneth theorem that we alluded to above (and whose proof we'll give in §5.7). Using these results we'll prove

Theorem 5.6.8. *The Thom class, T_Δ, in $H^n(X \times X)$ is given explicitly by the formula*

$$T_\Delta = \sum_{k+\ell=n} (-1)^\ell \sum_{i=1}^{d_k} \pi_1^\sharp \mu_i^k \cdot \pi_2^\sharp \nu_i^\rho\,. \tag{5.6.26}$$

Proof. We have to check that for every cohomology class, $c \in H^n(X \times X)$, the class, T_Δ, defined by (5.6.26) has the reproducing property

$$I_{X\times X}(T_\Delta \cdot c) = I_\Delta(\iota_\Delta^\sharp c) \tag{5.6.27}$$

where ι_Δ is the inclusion map of Δ into $X \times X$. However the map

$$\gamma_\Delta : X \to X \times X\,, \quad x \to (x, x)$$

is an orientation-preserving diffeomorphism of X onto Δ, so it suffices to show that

$$I_{X\times X}(T_\Delta \cdot c) = I_X(\gamma_\Delta^\sharp c) \tag{5.6.28}$$

and by Lemma 5.6.7 it suffices to verify (5.6.28) for c's of the form

$$c = \pi_1^\sharp \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k \,.$$

The product of this class with a typical summand of (5.6.26), for instance, the summand

$$(-1)^{\ell'} \pi_1^\sharp \mu_i^{k'} \cdot \pi_2^\sharp \nu_i^{\ell'}\,, \quad k'+\ell' = n\,, \tag{5.6.29}$$

is equal, up to sign to,

$$\pi_1^\sharp \mu_i^{k'} \cdot \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k \cdot \nu_i^{\ell'}\,.$$

Notice, however, that if $k \neq k'$ this product is zero: For $k < k'$, $k'+\ell$ is greater than $k+\ell$ and hence greater than n. Therefore

$$\mu_i^{k'} \cdot \nu_r^\ell \in H^{k'+\ell}(X)$$

is zero since X is of dimension n, and for $k > k'$, ℓ' is greater than ℓ and $\mu_s^k \cdot \nu_i^{\ell'}$ is zero for the same reason. Thus in taking the product of T_Δ with c we can ignore all terms in the sum except for the terms, $k' = k$ and $\ell' = \ell$. For these terms, the product of (5.6.29) with c is

$$(-1)^{k\ell} \pi_1^\sharp \mu_i^k \cdot \nu_r^\ell \cdot \pi_2^\sharp \mu_3^k \cdot \nu_i^\ell\,.$$

(Exercise: Check this. *Hint:* $(-1)^\ell(-1)^{\ell^2} = 1$.) Thus

$$T_\Delta \cdot c = (-1)^{k\ell} \sum_i \pi_1^\sharp \mu_i^k \cdot \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k \cdot \nu_i^\ell$$

and hence by Lemma 5.6.5 and (5.6.24)

$$\begin{aligned} I_{X\times X}(T_\Delta \cdot c) &= (-1)^{k\ell} \sum_i I_X(\mu_i^k \cdot \nu_r^\ell) I_X(\mu_s^k \cdot \nu_i^\ell) \\ &= (-1)^{k\ell} \sum_i \delta_{ir}\delta_{is} \\ &= (-1)^{k\ell} \delta_{rs}\,. \end{aligned}$$

On the other hand for $c = \pi_1^\sharp \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k$

$$\begin{aligned} \gamma_\Delta^\sharp c &= \gamma_\Delta^\sharp \pi_1^\sharp \nu_r^\ell \cdot \gamma_\Delta^\sharp \pi_2^\sharp \mu_s^k \\ &= (\pi_1 \cdot \gamma_\Delta)^\sharp \nu_r^\ell (\pi_2 \cdot \gamma_\Delta)^\sharp \mu_s^k \\ &= \nu_r^\ell \cdot \mu_s^k \end{aligned}$$

since

$$\pi_1 \cdot \nu_\Delta = \pi_2 \cdot \gamma_\Delta = id_X \,.$$

So

$$I_X(\gamma_\Delta^\sharp c) = I_X(\nu_r^\ell \cdot \mu_s^k) = (-1)^{k\ell} \delta_{rs}$$

by (5.6.24). Thus the two sides of (5.6.27) are equal.

□

We're now in position to compute $L(f)$, i.e., to compute the expression $I_X(\gamma_f^* T_\Delta)$ on the right hand side of (5.6.18). Since ν_i^ℓ, $i = 1, \ldots, d_\ell$ is a basis of $H^\ell(X)$ the linear mapping

$$f^\sharp : H^\ell(X) \to H^\ell(X) \tag{5.6.30}$$

can be described in terms of this basis by a matrix, $[f_{ij}^\ell]$ with the defining property

$$f^\sharp \nu_i^\ell = \sum f_{ji}^\ell \nu_j^\ell \,.$$

Thus by (5.6.26), (5.6.19) and (5.6.20)

$$\begin{aligned} \gamma_f^\sharp T_\Delta &= \gamma_f^\sharp (-1)^\ell \sum_{k+\ell=n} \sum_i \pi_1^\sharp u_i^k \cdot \pi_2^\sharp \nu_i^\ell \\ &= \sum (-1)^\ell \sum_i (\pi_1 \cdot \gamma_f)^\sharp \mu_i^k \cdot (\pi_2 \cdot \nu_f)^\sharp \nu_i^\ell \\ &= \sum (-1)^\ell \sum \mu_i^k \cdot f^\sharp \nu_i^\ell \\ &= \sum (-1)^\ell \sum f_{ji}^\ell \mu_i^k \cdot \nu_j^\ell \,. \end{aligned}$$

Thus by (5.6.24)

$$\begin{aligned} I_X(\gamma_f^\sharp T_\Delta) &= \sum (-1)^\ell \sum f_{ji}^\ell I_X(\mu_i^k \cdot \nu_j^\ell) \\ &= \sum (-1)^\ell \sum f_{ji}^\ell \delta_{ij} \\ &= \sum_{\ell=0}^n (-1)^\ell \left(\sum_i f_{ii}^\ell \right) \,. \end{aligned}$$

But $\sum_i f^\ell_{i,i}$ is just the trace of the linear mapping (5.6.30) (see exercise 12 below), so we end up with the following purely topological prescription of $L(f)$.

Theorem 5.6.9. *The Lefshetz number, $L(f)$ is the alternating sum*

$$\sum (-1)^\ell \operatorname{Trace}(f^\sharp)_\ell \tag{5.6.31}$$

where $\operatorname{Trace}(f^\sharp)_\ell$ *is the trace of the mapping*

$$f^\sharp : H^\ell(X) \to H^\ell(X)\,.$$

Exercises.

1. Show that if $f_0 : X \to X$ and $f_1 : X \to X$ are homotopic $\mathcal{C}^\infty$ mappings $L(f_0) = L(f_1)$.

2. (a) The Euler characteristic, $\chi(X)$, of X is defined to be the intersection number of the diagonal with itself in $X \times X$, i.e., the "self-intersection" number

$$I(\Delta, \Delta) = I_{X\times X}(T_\Delta, T_\Delta)\,.$$

Show that if a $\mathcal{C}^\infty$ map, $f : X \to X$ is homotopic to the identity, $L_f = \chi(X)$.

(b) Show that

$$\chi(X) = \sum_{\ell=0}^{n} (-1)^\ell \dim H^\ell(X)\,. \tag{5.6.32}$$

(c) Show that $\chi(X) = 0$ if n is odd.

3. (a) Let S^n be the unit n-sphere in $\mathbb{R}^{n+1}$. Show that if $g : S^n \to S^n$ is a $\mathcal{C}^\infty$ map

$$L(g) = 1 + (-1)^n (\deg)\,(\mathrm{g})\,.$$

(b) Conclude that if $\deg(g) \neq (-1)^{n+1}$, then g has to have a fixed point.

4. Let f be a $\mathcal{C}^\infty$ mapping of the closed unit ball, B^{n+1}, into itself and let $g : S^n \to S^n$ be the restriction of f to the boundary of B^{n+1}. Show that if $\deg(g) \neq (-1)^{n+1}$ then the fixed point of f predicted by Brouwer's theorem can be taken to be a point on the boundary of B^{n+1}.

5. (a) Show that if $g : S^n \to S^n$ is the antipodal map, $g(x) = -x$, then $\deg(g) = (-1)^{n+1}$.

(b) Conclude that the result in #4 is sharp. Show that the map

$$f : B^{n+1} \to B^{n+1}\,, \quad f(x) = -x\,,$$

has only one fixed point, namely the origin, and in particular has no fixed points on the boundary.

6. Let v be a vector field on X. Since X is compact, v generates a one-parameter group of diffeomorphisms

$$f_t : X \to X\,, \quad -\infty < t < \infty\,. \tag{5.6.33}$$

(a) Let $\sum_t$ be the set of fixed points of f_t. Show that this set contains the set of zeroes of v, i.e., the points, $p \in X$ where $v(p) = 0$.

(b) Suppose that for some t_0, f_{t_0} is Lefshetz. Show that for all t, f_t maps $\sum_{t_0}$ into itself.

(c) Show that for $|t| < \epsilon$, ϵ small, the points of $\sum_{t_0}$ are fixed points of f_t.

(d) Conclude that $\sum_{t_0}$ is equal to the set of zeroes of v.

(e) In particular, conclude that for all t the points of $\sum_{t_0}$ are fixed points of f_t.

7. (a) Let V be a finite dimensional vector space and

$$F(t) : V \to V\,,\ -\infty < t < \infty$$

a one-parameter group of linear maps of V onto itself. Let $A = \frac{dF}{dt}(0)$. Show that $F(t) = \exp tA$. (See §2.1, exercise 7.)

(b) Show that if $I - F(t_0) : V \to V$ is bijective for some t_0, then $A : V \to V$ is bijective. *Hint:* Show that if $A\mathrm{v} = 0$ for some $\mathrm{v} \in V - \{0\}$, $F(t)\mathrm{v} = \mathrm{v}$.

8. Let v be a vector field on X and let (5.6.33) be the one-parameter group of diffeomorphisms generated by v. If $v(p) = 0$ then by part (a) of exercise 6, p is a fixed point of f_t for all t.

(a) Show that

$$(df_t) : T_pX \to T_pX$$

is a one-parameter group of linear mappings of T_pX onto itself.

(b) Conclude from #7 that there exists a linear map

$$L_v(p) : T_pX \to T_pX \tag{5.6.34}$$

with the property

$$\exp tL_v(p) = (df_t)_p\,. \tag{5.6.35}$$

9. Suppose f_{t_0} is a Lefshetz map for some t_0. Let $a = t_0/N$ where N is a positive integer. Show that f_a is a Lefshetz map. *Hints:*

(a) Show that

$$f_{t_0} = f_a \circ \cdots \circ f_a = f_a^N$$

(i.e., f_a composed with itself N times).

(b) Show that if p is a fixed point of f_a, it is a fixed point of f_{t_0}.

(c) Conclude from exercise 6 that the fixed points of f_a are the zeroes of v.

(d) Show that if p is a fixed point of f_a,

$$(df_{t_0})_p = (df_a)_p^N\,.$$

(e) Conclude that if $(df_a)_p\mathrm{v} = \mathrm{v}$ for some $\mathrm{v} \in T_pX - \{0\}$, then $(df_{t_0})_p\mathrm{v} = \mathrm{v}$.

10. Show that for all t, $L(f_t) = \chi(X)$. *Hint:* Exercise 2.

11. (The Hopf theorem.) A vector field v on X is a *Lefshetz* vector field if for some t_0, f_{t_0} is a Lefshetz map.

(a) Show that if v is a Lefshetz vector field then it has a finite number of zeroes and for each zero, p, the linear map (5.6.34) is bijective.

(b) For a zero, p, of v let $\sigma_p(v) = +1$ if the map (5.6.34) is orientation-preserving and -1 if it's orientation-reversing. Show that

$$\chi(X) = \sum_{v(p)=0} \sigma_p(v) .$$

Hint: Apply the Lefshetz theorem to f_a, $a = t_0/N$, N large.

12. (The trace of a linear mapping: a quick review.)

For $A = [a_{i,j}]$ an $n \times n$ matrix define

$$\text{trace}\, A = \sum a_{i,i} .$$

(a) Show that if A and B are $n \times n$ matrices

$$\text{trace}\, AB = \text{trace}\, BA .$$

(b) Show that if B is an invertible $n \times n$ matrix

$$\text{trace}\, BAB^{-1} = \text{trace}\, A .$$

(c) Let V be and n-dimensional vector space and $L : V \to V$ a liner map. Fix a basis $\mathrm{v}_1, \dots, \mathrm{v}_n$ of V and define the trace of L to be the trace of A where A is the defining matrix for L in this basis, i.e.,

$$L\mathrm{v}_i = \sum a_{j,i}\mathrm{v}_j .$$

Show that this is an *intrinsic* definition not depending on the basis $\mathrm{v}_1, \dots, \mathrm{v}_n$.

5.7 The Künneth theorem

Let X be an n-dimensional manifold and Y an r-dimensional manifold, both of these manifolds having finite topology. Let

$$\pi : X \times Y \to X$$

be the projection map, $\pi(x, y) = x$ and

$$\rho : X \times Y \to Y$$

the projection map $(x, y) \to y$. Since X and Y have finite topology their cohomology groups are finite dimensional vector spaces. For $0 \le k \le n$ let

$$\mu_i^k, \quad 1 \le i \le \dim H^k(X),$$

be a basis of $H^k(X)$ and for $0 \le \ell \le r$ let

$$\nu_j^\ell, \quad 1 \le j \le \dim H^\ell(Y)$$

be a basis of $H^\ell(Y)$. Then for $k + \ell = m$ the product, $\pi^\sharp \mu_i^k \cdot \rho^\sharp \nu_j^\ell$, is in $H^m(X \times Y)$. The Künneth theorem asserts

Theorem 5.7.1. *The product manifold, $X \times Y$, has finite topology and hence the cohomology groups, $H^m(X \times Y)$ are finite dimensional. Moreover, the products over $k + \ell = m$*

$$\pi^\sharp \mu_i^k \cdot \rho^\sharp \nu_j^\ell, \ 0 \le i \le \dim H^k(X), \ 0 \le j \le \dim H^\ell(Y), \tag{5.7.1}$$

are a basis for the vector space $H^m(X \times Y)$.

The fact that $X \times Y$ has finite topology is easy to verify. If U_i, $i = 1, \ldots, M$, is a good cover of X and V_j, $j = 1, \ldots, N$, is a good cover of Y the products of these open sets, $U_i \times U_j$, $1 \le i \le M$, $1 \le j \le N$ is a good cover of $X \times Y$: For every multi-index, I, U_I is either empty or diffeomorphic to $\mathbb{R}^n$, and for every multi-index, J, V_J is either empty or diffeomorphic to $\mathbb{R}^r$, hence for any product multi-index (I, J), $U_I \times V_J$ is either empty or diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^r$. The tricky part of the proof is verifying that the products, (5.7.1) are a basis of $H^m(X \times Y)$, and to do this it will be helpful to state the theorem above in a form that avoids our choosing specified bases for $H^k(X)$ and $H^\ell(Y)$. To do so we'll need to generalize slightly the notion of a bilinear pairing between two vector space.

Definition 5.7.2. *Let V_1, V_2 and W be finite dimensional vector spaces. A map $B : V_1 \times V_2 \to W$ is a bilinear map if it is linear in each of its factors, i.e., for $\mathrm{v}_2 \in V_2$ the map*

$$\mathrm{v} \in V_1 \to B(\mathrm{v}_1, \mathrm{v}_2)$$

is a linear map of V_1 into W and for $\mathrm{v}_1 \in V_1$ so is the map

$$\mathrm{v} \in V_2 \to B(\mathrm{v}_1, \mathrm{v}).$$

It's clear that if B_1 and B_2 are bilinear maps of $V_1 \times V_2$ into W and λ_1 and λ_2 are real numbers the function

$$\lambda_1 B_1 + \lambda_2 B_2 : V_1 \times V_2 \to W$$

is also a bilinear map of $V_1 \times V_2$ into W, so the set of all bilinear maps of $V_1 \times V_2$ into W forms a vector space. In particular the set of all bilinear maps of $V_1 \times V_2$ into $\mathbb{R}$ is a vector space, and since this vector space will play an essential role in our intrinsic formulation of the Künneth theorem, we'll give it a name. We'll call it the *tensor product* of V_1^* and V_2^* and denote it by $V_1^* \otimes V_2^*$. To explain where this terminology comes from we note that if ℓ_1 and ℓ_2 are vectors in V_1^* and V_2^* then one can define a bilinear map

$$\ell_1 \otimes \ell_2 : V_1 \times V_2 \to \mathbb{R} \tag{5.7.2}$$

by setting $(\ell_1 \otimes \ell_2)(\mathrm{v}_1, \mathrm{v}_2) = \ell_1(\mathrm{v}_1)\ell_2(\mathrm{v}_2)$. In other words one has a tensor product map:

$$V_1^* \times V_2^* \to V_1^* \otimes V_2^* \tag{5.7.3}$$

mapping (ℓ_1, ℓ_2) to $\ell_1 \otimes \ell_2$. We leave for you to check that this is a bilinear map of $V_1^* \times V_2^*$ into $V_1^* \otimes V_2^*$ and to check as well

Proposition 5.7.3. *If ℓ_i^1, $i = 1, \dots, m$ is a basis of V_1^* and ℓ_j^2, $j = 1, \dots, n$ is a basis of V_2^* then $\ell_i^1 \otimes \ell_j^2$, $1 \le i \le m$, $1 \le j \le n$, is a basis of $V_1^* \otimes V_2^*$.*

Hint: If V_1 and V_2 are the same vector space you can find a proof of this in §1.3 and the proof is basically the same if they're different vector spaces.

Corollary 5.7.4. *The dimension of $V_1^* \otimes V_2^*$ is equal to the dimension of V_1^* times the dimension of V_2^*.*

We'll now perform some slightly devious maneuvers with "duality" operations. First note that for any finite dimensional vector space, V, the pairing

$$V \times V^* \to \mathbb{R}\,, \quad (\mathrm{v}, \ell) \to \ell(\mathrm{v}) \tag{5.7.4}$$

is a non-singular bilinear pairing, so, as we explained in §5.4 it gives rise to a bijective linear mapping

$$V \to (V^*)^*\,. \tag{5.7.5}$$

Next note that if

$$L : V_1 \times V_2 \to W \tag{5.7.6}$$

is a bilinear mapping and $\ell : W \to \mathbb{R}$ a linear mapping (i.e., an element of W^*), then the composition of ℓ and L is a bilinear mapping

$$\ell \circ L : V_1 \times V_2 \to \mathbb{R}$$

and hence by definition an element of $V_1^* \otimes V_2^*$. Thus from the bilinear mapping (5.7.6) we get a *linear* mapping

$$L^\sharp : W^* \to V_1^* \otimes V_2^* \,. \tag{5.7.7}$$

We'll now define a notion of tensor product for the vector spaces V_1 and V_2 themselves.

Definition 5.7.5. *The vector space, $V_1 \otimes V_2$ is the vector space dual of $V_1^* \otimes V_2^*$, i.e., is the space*

$$V_1 \otimes V_2 = (V_1^* \otimes V_2^*)^* \,. \tag{5.7.8}$$

One implication of (5.7.8) is that there is a natural bilinear map

$$V_1 \times V_2 \to V_1 \otimes V_2 \,. \tag{5.7.9}$$

(In (5.7.3) replace V_i by V_i^* and note that by (5.7.5) $(V_i^*)^* = V_i$.) Another is the following:

Proposition 5.7.6. *Let L be a bilinear map of $V_1 \times V_2$ into W. Then there exists a unique linear map*

$$L^\# : V_1 \otimes V_2 \to W \tag{5.7.10}$$

with the property

$$L^\#(\mathrm{v}_1 \otimes \mathrm{v}_2) = L(\mathrm{v}_1, \mathrm{v}_2) \tag{5.7.11}$$

where $\mathrm{v}_1 \otimes \mathrm{v}_2$ is the image of $(\mathrm{v}_1, \mathrm{v}_2)$ with respect to (5.7.9).

Proof. Let $L^\#$ be the transpose of the map $L^\sharp$ in (5.7.7) and note that by (5.7.5) $(W^*)^* = W$.

$\square$

Notice that by Proposition 5.7.6 the property (5.7.11) is the *defining* property of $L^{\#}$, it uniquely determines this map. (This is in fact the whole point of the tensor product construction. Its purpose is to convert bilinear objects into linear objects.)

After this brief digression (into an area of mathematics which some mathematicians unkindly refer to as "abstract nonsense") let's come back to our motive for this digression: an intrinsic formulation of the Künneth theorem. As above let X and Y be manifolds of dimension n and r, respectively, both having finite topology. For $k+\ell=m$ one has a bilinear map

$$H^k(X) \times H^\ell(Y) \to H^m(X \times Y)$$

mapping (c_1, c_2) to $\pi^* c_1 \cdot \rho^* c_2$, and hence by Proposition 5.7.6 a *linear* map

$$H^k(X) \otimes H^\ell(Y) \to H^m(X \times Y)\,. \tag{5.7.12}$$

Let

$$H_1^m(X \times Y) = \sum_{k+\ell=m} H^k(X) \otimes H^\ell(Y)\,.$$

The maps (5.7.12) can be combined into a single linear map

$$H_1^m(X \times Y) \to H^m(X \times Y) \tag{5.7.13}$$

and our intrinsic version of the Künneth theorem asserts

Theorem 5.7.7. *The map (5.7.13) is bijective.*

Here is a sketch of how to prove this. (Filling in the details will be left as a series of exercises.) Let U be an open subset of X which has finite topology and let

$$\mathcal{H}_1^m(U) = \sum_{k+\ell=m} H^k(U) \otimes H^\ell(Y)$$

and

$$\mathcal{H}_2^m(U) = H^m(U \times Y)\,.$$

As we've just seen there's a Künneth map

$$\kappa : \mathcal{H}_1^m(U) \to \mathcal{H}_2^m(U)\,.$$

Exercises.

1. Let U_1 and U_2 be open subsets of X, both having finite topology, and let $U = U_1 \cup U_2$. Show that there is a long exact sequence:

$$\xrightarrow{\delta} \mathcal{H}_1^m(U) \longrightarrow \mathcal{H}_1^m(U_1) \oplus \mathcal{H}_1^m(U_2) \longrightarrow \mathcal{H}_1^m(U_1 \cap U_2) \xrightarrow{\delta} \mathcal{H}_1^{m+1}(U) \longrightarrow$$

Hint: Take the usual Mayer–Victoris sequence:

$$\xrightarrow{\delta} H^k(U) \longrightarrow H^k(U_1) \oplus H^k(U_2) \longrightarrow H^k(U_1 \cap U_2) \xrightarrow{\delta} H^{k+1}(U) \longrightarrow$$

tensor each term in this sequence with $H^\ell(Y)$ and sum over $k + \ell = m$.

2. Show that for $\mathcal{H}_2$ there is a similar sequence. *Hint:* Apply Mayer–Victoris to the open subsets $U_1 \times Y$ and $U_2 \times Y$ of M.

3. Show that the diagram below commutes. (This looks hard but is actually very easy: just write down the definition of each arrow in the language of forms.)

$$\begin{array}{ccccccccc}
\xrightarrow{\delta} & \mathcal{H}_2^m(U) & \longrightarrow & \mathcal{H}_2^m(U_1) \oplus \mathcal{H}_2^m(U_2) & \longrightarrow & \mathcal{H}_2^m(U_1 \cap U_2) & \xrightarrow{\delta} & \mathcal{H}_2^{m+1}(U) & \longrightarrow \\
 & k\uparrow & & k\uparrow & & k\uparrow & & k\uparrow & \\
\xrightarrow{\delta} & \mathcal{H}_1^m(U) & \longrightarrow & \mathcal{H}_1^m(U_1) \oplus \mathcal{H}_1^m(U_2) & \longrightarrow & \mathcal{H}_1^m(U_1 \cap U_2) & \xrightarrow{\delta} & \mathcal{H}_1^{m+1}(U) & \longrightarrow
\end{array}$$

4. Conclude from Exercise 3 that if the Künneth map is bijective for U_1, U_2 and $U_1 \cap U_2$ it is bijective for U.

5. Prove the Künneth theorem by induction on the number of open sets in a good cover of X. To get the induction started, note that

$$H^k(X \times Y) \cong H^k(Y)$$

if $X = \mathbb{R}^n$. (See §5.3, exercise 11.)