Appendix C Good covers and convexity theorems.

Let X be an n dimensional submanifold of $\mathbb{R}^N$. Our goal in this appendix is to prove that X admits a good cover. To do so we'll need some basic facts about "convexity". Let U be an open set in $\mathbb{R}^n$ and ϕ a C^∞ function on U.
Definition ϕ is convex if the matrix

$$\text{(C1)} \qquad \left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}(p)\right], \qquad 1 \le i, j \le n$$

is positive definite for all $p \in U$

Suppose now that U itself is convex and that ϕ is a convex function on U which has as image the half open interval $[0, a)$ and is a proper mapping of U onto $[0, a)$ (In other words for $0 \le c \le a$ the set, $\phi \le c$ is compact.)

Theorem 5.8.4. *For $0 < c < q$ the set, $U_c : \phi < C$ is convex*

Proof. For $p, q \in Bd\, U_c$, $p \neq q$ let $f(t) = \phi\left(((1-t)p + tq)\right), 0 \le t \le 1$.
We claim

$$\text{(C2)} \qquad \frac{d^2 f}{dt^2}(t) > 0$$

and

$$\text{(C3)} \qquad \frac{df}{dt}(0) < 0 < \frac{\partial f}{\partial t}(1)$$

□

Proof. To prove (C2) we note that

$$\frac{d^2 f}{dt^2}(t) = \sum \frac{\partial^2 Q}{\partial x_i \partial x_j}\left(((1-t)p + tq)\right)(p_i - q_i)(p_j - q_j)$$

and that the right hand side is positive because of the positive definiteness of (C1). □

To prove (C3) we note that $\dfrac{df}{dt}$ is strictly increasing by (C2). Hence "$\dfrac{df}{dt}(0) > 0$" would imply "$\dfrac{df}{dt}(t) > 0$" for all t and hence would imply "f strictly increasing". But $f(0) = f(1) = \phi(p) = \phi(q) = c$.

(A similar argument shows that $\frac{df}{dt}(1) > 0$.)

Since $\frac{df}{dt}$ is strictly increasing it follows from (C3) that $\frac{df}{dt}(t_0) = 0$ for some $0 < t_0 < 1$ and that $\frac{df}{dt}$ is less than 0 on the interval, $0 < t < t_0$, and greater than 0 on the interval $t_0 < t < 1$. Therefore, since $f(0) = f(1) = c$, $f(t) < c$ for all $0 < t < 1$; so $\phi < c$ on the line segment, $(1-t)p + tq$, $0 < t < 1$. Hence U_c is convex.

Coming back to the manifold, $X \subseteq \mathbb{R}^N$ let p be a point on X and let $L_p = T_pX$. We will denote by π_p the orthogonal projection

$$\pi_p : X \to L_p + p,$$

i.e. for $q \in X$ and $x = \pi_p(q)$ $q - x$ is orthogonal to L_p (it's easy to see that π_p is defined by this condition.) We will prove that this projection has the following convexity property.

Theorem 5.8.5. *There exists a positive constant, $c(p)$, such that if q is any point on X satisfying $|p-q|^2 < c(p)$ and c is less than $c(p)$ the set*

(C4) $$U(q,c) = \left\{q' \in X, |q'-q| < c\right\}$$

gets mapped diffeomorphically by π_p onto a convex open set in $L_p + p$.

Proof. We can without loss of generality assume that $p = 0$ and that $L = \mathbb{R}^n = \mathbb{R}^n \times \{0\}$ in $\mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^N$, where $k = N - n$. Thus X can be described, locally over a convex neighborhood, U, of the origin in $\mathbb{R}^n$ as the graph of a C^∞ function, $f : (U,0) \to \left(\mathbb{R}^k, 0\right)$ and since $\mathbb{R}^n \times \{0\}$ is tangent to X at $\{0\}$

(C5) $$\frac{\partial f}{\partial x_i}(0) = 0, \qquad i = 1, \ldots, n$$

□

Given $q = (x_q, f(x_q)) \in X$ let $\phi_q : u \to \mathbb{R}$ be the function

(C6) $$\phi_q(x) = |x - x_q|^2 + |f(x) - f(x_q)|^2$$

Then, by (C4), $\pi_p(U(q,c))$ is just the set, $\phi_q < c$; so to prove the theorem it suffices to prove that if $c(p)$ is sufficiently small and $c < c(p)$ this set is convex. However at $q = 0$,

$$\frac{\partial^2 Q_q}{\partial x_i \partial x_j}(x_q) = \delta_{ij}$$

by (C5) and (C6) and hence by continuity ϕ_q is convex on the set $|x|^2 < \delta$ provided that that δ and $|q|$ are sufficiently small. Hence if $c(p)$ is sufficiently small, $\pi_p(U(q,c))$ is convex for $c < c(p)$.

We will now use theorem 2 to prove that X admits a good cover: For every $p \in X$ let $\epsilon(p) = \sqrt{c(p)}/3$ and let U_p be the set of points, $q \in X$, with $|p - q| < \epsilon(p)$. We claim

Theorem 5.8.6. *The U_p's are a good cover of X.*

Proof. Suppose that the intersection,

$$U_{p_1} \cap \ldots \cap U_{p_k} \tag{C7}$$

is non-empty and that

$$\epsilon(p_1) \geq \epsilon(p_2) \geq \cdots \geq \epsilon(p_k)$$

□

Then, if p is a point in this intersection, $|p - p_i|$ and $|p - p_1|$ are less than $\epsilon(p_1)$ and hence $|p_i - p_1|$ is less than $2\epsilon(p_1)$. Moreover, if q is in U_{p_i}, $|q - p_i|$ is less than $\epsilon(p_1)$, so $|q - p_1|$ is less than $3\epsilon(p_1)$ and hence $|q - p_1|^2$ is less than $c(p_1)$. Therefore the set U_{p_i} is contained in $U(p_1, c(p_1))$ and consequently by theorem 2 is mapped diffeomorphically onto a convex open subset of $L_{p_1} + p_1$ by the projection π_{p_1}. Consequently the intersection, (C7) is mapped diffeomorphically onto a convex open subset of $L_{p_1} + p_1$ by π_{p_1}.