p. 212, Problem 12: Suppose we had a closed minimal surface $S \subset \mathbb{R}^3$. Then there exists a point $p \in S$ which is of maximal distance from the origin. We claim that K_p, the Gaussian curvature of S at p, is positive. This is a contradiction, since for a minimal surface, we have $H = \frac{1}{2}(k_1 + k_2) = 0$, which implies $K = k_1 k_2 = -k_1^2 \leq 0$ identically.

To prove the claim, let $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ be any parametrization of S near p. Then $f(u,v) = \frac{1}{2}\|\mathbf{x}(u,v)\|^2$ achieves a maximum at p. It follows that at p, we have

$$\partial_u f = \mathbf{x} \cdot \mathbf{x}_u = 0$$
$$\partial_v f = \mathbf{x} \cdot \mathbf{x}_v = 0.$$

Consequently, at p, $\mathbf{x}$ is perpendicular to the tangent plane spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$, i.e. $\mathbf{x}(0,0)$ is parallel to the normal N_p at p. If we differentiate the above again, since f is a maximum, we get

$$0 > \partial_{uu} f = \mathbf{x}_u \cdot \mathbf{x}_u + \mathbf{x} \cdot \mathbf{x}_{uu}.$$
$$0 > \partial_{vvf} = \mathbf{x}_v \cdot \mathbf{x}_v + \mathbf{x} \cdot \mathbf{x}_{vv}.$$

This implies

$$\mathbf{x} \cdot \mathbf{x}_{uu} < -\mathbf{x}_u \cdot \mathbf{x}_u < 0$$
$$\mathbf{x} \cdot \mathbf{x}_{vv} < -\mathbf{x}_v \cdot \mathbf{x}_v < 0.$$

Since at p, $\mathbf{x} = N_p$, the left-hand side of the above quantities are precisely $II_p(\mathbf{x}_u, \mathbf{x}_u)$ and $II_p(\mathbf{x}_v, \mathbf{x}_v)$, respectively, both of which are negative. In particular, they have the same sign. This implies $K_p > 0$.