18.950 - Pset #7

November 9, 2010

(22a) We want to compute $\frac{d}{dt}^2(h \circ \alpha)$, where $\alpha'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$. To minimize notation, I will not always be explicit in my notation concerning the dependence on t in the below (it should be clear that everything depends on t). First, we have

$$\begin{aligned}\frac{d}{dt}(h \circ \alpha) &= (dh)_{\alpha(t)}(\alpha'(t)) \\ &= (h_u \circ \alpha(t))u' + (h_v \circ \alpha(t))v'\end{aligned}$$

Note that $(dh)_{\alpha(t)}$ is the differential form dh evaluated at the point $\alpha(t)$, which takes as input the tangent vector $\alpha'(t)$. The terms $h_u \circ \alpha(t)$ and $h_v \circ \alpha(t)$ are *functions*, which get multiplied with the components u' and v' of the tangent vector $\alpha'(t)$. Do not confuse a differential form dh with its component functions h_u and h_v. Differentiating again,

$$\begin{aligned}\frac{d^2}{dt^2}(h \circ \alpha) &= \frac{d}{dt}\Big((h_u \circ \alpha(t))u'\Big) + \frac{d}{dt}\Big((h_v \circ \alpha(t))v'\Big) \\ &= (h_{uu} \circ \alpha)u'^2 + (h_{uv} \circ \alpha)u'v' + (h_u \circ \alpha)u'' \\ &\quad + (h_{vu} \circ \alpha)v'u' + (h_{vv} \circ \alpha)v'v' + (h_v \circ \alpha)v''.\end{aligned}$$

When we evaluate at $t = 0$, where $\alpha(0) = p$, then since $(dh)_p = 0$, then $h_u(p) = h_v(p) = 0$. So all the terms with two derivatives of u and v vanish and so the above reduces to

$$\frac{d^2}{dt^2}(h \circ \alpha)|_{t=0} = h_{uu}(p)(u')^2 + 2h_{uv}(p)u'v' + h_{vv}(p)(v')^2.$$

Once you become comfortable with the above objects and their identities, you rarely will want to keep track of all such notation as above (which I only did for pedagogical purposes).

(23(c) (This seemed like a bit of an unfair question since to solve it completely requires a bit of extra knowledge to deal with finicky complications in the most general case.) Note that B is the complement of the union of the four surfaces $S_{1,\pm}$ and $S_{2,\pm}$, defined by

$$\begin{aligned}S_{1,\pm} &= \{p \pm (k_1(p))^{-1}N_p : p \in S, k_1(p) \neq 0\} \\ S_{2,\pm} &= \{p \pm (k_1)^{-2}N_p : p \in S, k_2(p) \neq 0\}\end{aligned}$$

Indeed, h_r can only have nondegenerate critical points if r lies in the above surfaces by part (a) and (b). It is easy to see that these surfaces are all closed (since S is closed), so that the complement of their union B is open. The tricky part is showing that B is dense. For this, we want to show that the union of the above four surfaces cannot contain an open ball. For this, you can convince yourselves that these are "nice" surfaces, so that whenever $k_1(p)$ and $k_2(p)$ varying smoothly, these surfaces will be locally regular surfaces. In general, these surfaces will self-intersect. Moreover, $k_1(p)$ and $k_2(p)$ may only vary continuously at points where they become equal in value (since they contain a square root in their formula), but at least they are *Holder continuous*. From this, one can conclude that the Lebesgue measure of these four surfaces is zero, so that their union cannot contain a ball (since a ball has positive measure). This shows that the complement of these surfaces, B, is dense.

Note: There are well-known continuous space-filling curves which fill up a square or cube. Thus, it is possible to map a surface (only in a very bad way) continuously onto a ball. The point is to show that the above surfaces do not do this, since principal curvatures and normal curvatures vary in a mild way (Holder continuous is enough).

p. 187, 7(a) One direction is clear. If w is differentiable and f is differentiable, then so is $w(f)$ by the chain-rule. The other direction needs work. Let $\phi : \tilde{U} \to U \subset S$ be a local parametrization of S, where $\tilde{U} \subset \mathbb{R}^2$ has coordinates (u, v). Define $f_1 : U \to \mathbb{R}$ so that $(f_1 \circ \phi)(u, v) = u$. Likewise, define $f_2 : U \to \mathbb{R}$ so that $(f_2 \circ \phi)(u, v) = v$. In other words, f_1 and f_2 are the coordinate functions on $\tilde{U}$ when we pullback by ϕ. If $w = a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v$, then

$$\begin{aligned} w(f_1) &= (df_1)(a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v) \\ &= d(f_1 \circ \phi)(a(u, v)\partial_u + b(u, v)\partial_v) \\ &= a(u, v) \end{aligned}$$

since $(f_1 \circ \phi) = u$ and so $(f_1 \circ \phi)_u = 1$ and $(f_1 \circ \phi)_v = 0$. Similarly, $w(f_1) = b(u, v)$. Thus, by hypothesis, if $w(f)$ is differentiable for all differentiable f, then in particular, $w(f_1) = a$ and $w(f_2) = b$ are differentiable since f_1 and f_2 are differentiable. This shows that the components of a and b of w are differentiable, hence w is differentiable.