18.950 - Pset #6

November 2, 2010

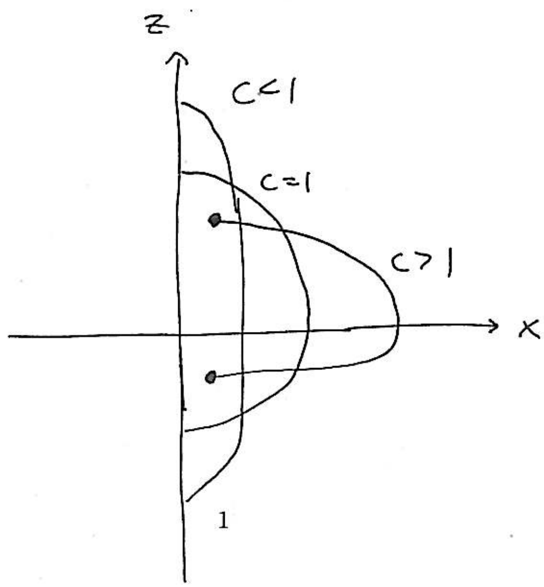
(7a) The equation $\varphi'' + K\varphi = \varphi$ is just rearrangement of Equation 9 on p. 162. One can solve for $\psi(v)$ in terms of $\phi(v)$ using the relation $(\phi')^2 + (\psi')^2 = 1$, thus obtaining $\psi(v) = \int \sqrt{1-(\varphi')^2}dv$, where the limits of integration are such that the integral makes sense.

(7b) From (7a), we need to solve the equation $\varphi'' + \varphi = 0$. The general solution to this equation is $\varphi(v) = A\cos v + B\sin v$. Since the surface in question is given parametrically by $S(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$, in order to intersect the xy-axis perpendicularly, we need the tangent vector dS/dv to be parallel to the z-axis whenever S intersects the xy-plane. Observe that since ψ is monotone in v, this happens for a unique $v = v_0$, i.e. the v_0 for which $\psi(v_0) = 0$. Without loss of generality, let $v_0 = 0$. The surface thus intersects the xy-plane in the curve $S(u,0)$, $0 \le u \le 2\pi$.

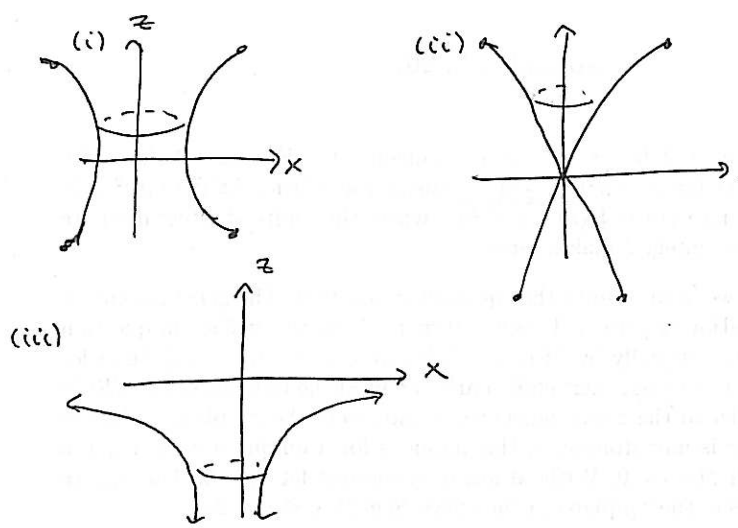
So we require $(dS/dv)(u,0) = 0$ to be zero for all $0 \le u \le 2\pi$. Since this must hold for all u, it follows that we need $\varphi'(0) = 0$. It follows that $\varphi(v) = C\cos v$ for some C. It now follows that

$$\psi(v) = \int_0^v \sqrt{1 - C^2\sin^2(w)}dw.$$

In order for the integral defining ψ to make sense, we need v such that $-1/C \le \sin v \le 1/C$. For $C \le 1$, this poses no restriction v, but for $C > 1$, then v cannot be arbitrary (in particular, $\psi(v)$ never achieves the value of zero). We have the following sketches:



(7c) With $K = -1$, we have to solve $\varphi'' - \varphi = 0$. The general solution is $\varphi(v) = Ae^v + Be^{-v}$. There are three cases: $A, B > 0$, A and B have opposite sign, or exactly one of A or B is zero. In either of these cases, by shifting and/or reflecting v (i.e. replacing v with $\pm v + v_0$ for some constant v_0), we can assume we that φ is of the form $C \cosh v$, $C \sin v$, or Ce^v, respectively. We have the following sketches:



(7d) There are two ways to see this. One can check the defining property of the pseudosphere in Exercise 6(a) (using $(\varphi')^2 + (\psi')^2 \equiv 1$). Alternatively, since the pseudosphere has constant $K = -1$, it has to be one of the above three cases, and from the sketches, we see that only in the third case is a surface infinite in extent.

(7e) In this case, $\varphi'' = 0$, so $\varphi(v) = av + b$, with $|a| \leq 1$. (i) If $a = 0$, then $b \neq 0$ necessarily for S to be a surface. In this case, we get a cylinder. (ii) If $|a| < 1$, then $\psi(v) = \sqrt{1 - a^2}v$ is nonzero and we get a cone. (iii) If $|a| = 1$, then $\psi \equiv 0$, and we get a plane.