17 The functional equation

In the previous lecture we proved that the Riemann zeta function $\zeta(s)$ has an Euler product and an analytic continuation to the right half-plane $\mathrm{Re}(s) > 0$. In this lecture we complete the picture by deriving a *functional equation* that relates the values of $\zeta(s)$ to those of $\zeta(1-s)$. This will then also allow us to extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ that is holomorphic except for a simple pole at $s = 1$.

17.1 Fourier transforms and Poisson summation

A key tool we will use to derive the functional equation is the *Poisson summation formula*, a result from harmonic analysis that we now recall.

Definition 17.1. A *Schwartz function* on $\mathbb{R}$ is a complex-valued C^∞ function $f\colon \mathbb{R} \to \mathbb{C}$ that decays rapidly to zero: for all $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\sup_{x\in\mathbb{R}} \left| x^m f^{(n)}(x) \right| < \infty,$$

where $f^{(n)}$ denotes the nth derivative of f. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ of all Schwartz functions on $\mathbb{R}$ is a (non-unital) $\mathbb{C}$-algebra of infinite dimension.

Example 17.2. All compactly supported C^∞ functions are Schwartz functions, as is the *Gaussian* function $g(x) := e^{-\pi x^2}$. Non-examples include functions that do not tend to zero as $x \to \pm\infty$ (such as polynomials), and functions like $(1 + x^{2n})^{-1}$ and $e^{-x^2}\sin(e^{x^2})$ that either do not tend to zero quickly enough, or have derivatives that do not tend to zero as $x \to \pm\infty$.

Remark 17.3. For any $p \in \mathbb{R}_{\geq 1}$, the Schwartz space $\mathcal{S}(\mathbb{R})$ is contained in the space $L^p(\mathbb{R})$ of functions on $f\colon \mathbb{R} \to \mathbb{C}$ for which the Lebesgue integral $\int_{\mathbb{R}} |f(x)|^p dx$ exists. The space $L^p(\mathbb{R})$ is a complete normed $\mathbb{C}$-vector space under the L^p-norm $\|f\|_p := (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$, and is thus a Banach space. The Schwartz space $\mathcal{S}(\mathbb{R})$ is not complete under the L^p-norm, but it is dense in $L^p(\mathbb{R})$ (in the subspace topology). One can equip the Schwartz space with a translation-invariant metric of its own under which it is a complete metric space (and thus a Fréchet space, since it is also locally convex), but the topology of $\mathcal{S}(\mathbb{R})$ will not concern us here. Similar comments apply to $\mathcal{S}(\mathbb{R}^n)$.

It follows immediately from the definition and standard properties of the derivative that the Schwartz space $\mathcal{S}(\mathbb{R})$ is closed under differentiation, multiplication by polynomials, and linear change of variable. It is also closed under *convolution*: for any $f, g \in \mathcal{S}(\mathbb{R})$ the function

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)dy$$

is also an element of $\mathcal{S}(\mathbb{R})$. Convolution is commutative, associative, and bilinear.

Definition 17.4. The *Fourier transform* of a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ is the function

$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e^{-2\pi i x y}dx,$$

which is also a Schwartz function [1, Thm. 5.1.3]. We can recover $f(x)$ from $\hat{f}(y)$ via the inverse transform

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{+2\pi i x y} dy;$$

see [1, Thm. 5.1.9] for a proof of this fact. The maps $f \mapsto \hat{f}$ and $\hat{f} \mapsto f$ are thus inverse linear operators on $\mathcal{S}(\mathbb{R})$ (they are also continuous in the metric topology of $\mathcal{S}(\mathbb{R})$ and thus homeomorphisms).

Remark 17.5. The invertibility of the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R})$ is a key motivation for its definition. For functions in $L^1(\mathbb{R})$ (the largest space of functions for which our definition of the Fourier transform makes sense), the Fourier transform of a smooth function decays rapidly to zero, and the Fourier transform of a function that decays rapidly to zero is smooth; this leads one to consider the subspace $\mathcal{S}(\mathbb{R})$ of smooth functions that decay rapidly to zero. One can show that $\mathcal{S}(\mathbb{R})$ is the largest subspace of $L^1(\mathbb{R})$ closed under multiplication by polynomials on which the Fourier transform is invertible.[1]

The Fourier transform changes convolutions into products, and vice versa. We have

$$\widehat{f * g} = \hat{f}\hat{g} \qquad \text{and} \qquad \widehat{fg} = \hat{f} * \hat{g},$$

for all $f, g \in \mathcal{S}(\mathbb{R})$ (see Problem Set 8). One can thus view the Fourier transform as an isomorphism of (non-unital) $\mathbb{C}$-algebras that sends $(\mathcal{S}(\mathbb{R}), +, \times)$ to $(\mathcal{S}(\mathbb{R}), +, *)$.

Lemma 17.6. *For all $a \in \mathbb{R}_{>0}$ and $f \in \mathcal{S}(\mathbb{R})$, we have $\widehat{f(ax)}(y) = \frac{1}{a}\hat{f}(\frac{y}{a})$.*

Proof. Applying the substitution $t = ax$ yields

$$\widehat{f(ax)}(y) = \int_{\mathbb{R}} f(ax) e^{-2\pi i x y} dx = \frac{1}{a}\int_{\mathbb{R}} f(t) e^{-2\pi i t y/a} dt = \frac{1}{a}\hat{f}\left(\frac{y}{a}\right). \qquad \square$$

Lemma 17.7. *For $f \in \mathcal{S}(\mathbb{R})$ we have $\frac{d}{dy}\hat{f}(y) = -2\pi i\, \widehat{xf(x)}(y)$ and $\widehat{\frac{d}{dx}f(x)}(y) = 2\pi i y \hat{f}(y)$.*

Proof. Noting that $xf \in \mathcal{S}(\mathbb{R})$, the first identity follows from

$$\tfrac{d}{dy}\hat{f}(y) = \tfrac{d}{dy}\left(\int_{\mathbb{R}} f(x) e^{-2\pi i x y} dx\right) = \int_{\mathbb{R}} f(x)(-2\pi i x) e^{-2\pi i x y} dx = -2\pi i\, \widehat{xf(x)}(y),$$

since we may differentiate under the integral via dominated convergence. For the second, we note that $\lim_{x\to\pm\infty} f(x) = 0$, so integration by parts yields

$$\widehat{\tfrac{d}{dx}f(x)}(y) = \int_{\mathbb{R}} f'(x) e^{-2\pi i x y} dx = 0 - \int_{\mathbb{R}} f(x)(-2\pi i y) e^{-2\pi i x y} dx = 2\pi i y \hat{f}(y). \qquad \square$$

The Fourier transform is compatible with the inner product $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}dx$ on $L^2(\mathbb{R})$ (which contains $\mathcal{S}(\mathbb{R})$). Indeed, we can easily derive Parseval's identity:

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(y)\overline{g(x)}e^{+2\pi i x y} dx dy = \int_{\mathbb{R}} \hat{f}(y)\overline{\hat{g}(y)}dy = \langle \hat{f}, \hat{g}\rangle,$$

which when applied to $g = f$ yields Plancherel's identity:

$$\|f\|_2^2 = \langle f, f\rangle = \langle \hat{f}, \hat{f}\rangle = \|\hat{f}\|_2^2,$$

where $\|f\|_2 = (\int_{\mathbb{R}} |f(x)|^2 dx)^{1/2}$ is the L^2-norm. For number-theoretic applications there is an analogous result due to Poisson.

[1] I thank Keith Conrad and Terry Tao for clarifying this point.

Theorem 17.8 (Poisson Summation Formula)**.** *For all $f \in \mathcal{S}(\mathbb{R})$ we have the identity*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Proof. We first note that both sums are well defined; the rapid decay property of Schwartz functions guarantees absolute convergence. Let $F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$. Then F is a periodic C^∞-function, so it has a Fourier series expansion

$$F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x},$$

with Fourier coefficients

$$c_n = \int_0^1 F(t)e^{-2\pi i n t}dt = \int_0^1 \sum_{m\in\mathbb{Z}} f(t+m)e^{-2\pi i n t}dt = \int_{\mathbb{R}} f(t)e^{-2\pi i n t}dt = \hat{f}(n).$$

We then note that

$$\sum_{n\in\mathbb{Z}} f(n) = F(0) = \sum_{n\in\mathbb{Z}} c_n = \sum_{n\in\mathbb{Z}} \hat{f}(n). \qquad \square$$

Finally, we note that the Gaussian function $e^{-\pi x^2}$ is its own Fourier transform.

Lemma 17.9. *Let $g(x) := e^{-\pi x^2}$. Then $\hat{g}(y) = g(y)$.*

Proof. The function $g(x)$ satisfies the first order ordinary differential equation

$$g' + 2\pi x g = 0, \tag{1}$$

with initial value $g(0) = 1$. Multiplying both sides by $-i$ and taking Fourier transforms yields

$$-i(\widehat{g'} + 2\pi\widehat{xg}) = -i(2\pi i x\hat{g} + i\hat{g}') = \hat{g}' + 2\pi x\hat{g} = 0,$$

via Lemma 17.7. So $\hat{g}$ also satisfies (1), and $\hat{g}(0) = \int_{\mathbb{R}} e^{-\pi x^2}dx = 1$, so $\hat{g} = g$. $\square$

17.1.1 Jacobi's theta function

We now define the *theta function*[2]

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}.$$

The sum is absolutely convergent for $\operatorname{im}\tau > 0$ and thus defines a holomorphic function on the upper half plane. It is easy to see that $\Theta(\tau)$ is periodic modulo 2, that is,

$$\Theta(\tau+2) = \Theta(\tau),$$

but it it also satisfies another functional equation.

Lemma 17.10. *For all $a \in \mathbb{R}_{>0}$ we have $\Theta(ia) = \Theta(i/a)/\sqrt{a}$.*

Proof. Put $g(x) := e^{-\pi x^2}$ and $h(x) := g(\sqrt{a}x) = e^{-\pi x^2 a}$. Lemmas 17.6 and 17.9 imply

$$\hat{h}(y) = \widehat{g(\sqrt{a}x)}(y) = \hat{g}\big(y/\sqrt{a}\big)/\sqrt{a} = g\big(y/\sqrt{a}\big)/\sqrt{a}.$$

Plugging $\tau = ia$ into $\Theta(\tau)$ and applying Poisson summation (Theorem 17.8) yields

$$\Theta(ia) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 a} = \sum_{n\in\mathbb{Z}} h(n) = \sum_{n\in\mathbb{Z}} \hat{h}(n) = \sum_{n\in\mathbb{Z}} g\big(n/\sqrt{a}\big)/\sqrt{a} = \Theta(i/a)/\sqrt{a}. \qquad \square$$

[2]The function $\Theta(\tau)$ we define here is a special case of one of four parameterized families of theta functions $\Theta_i(z:\tau)$ originally defined by Jacobi for $i = 0,1,2,3$, which play an important role in the theory of elliptic functions and modular forms; in terms of Jacobi's notation, $\Theta(\tau) = \Theta_3(0;\tau)$.

17.1.2 Euler's gamma function

You are probably familiar with the gamma function $\Gamma(s)$, which plays a key role in the functional equation of not only the Riemann zeta function but many of the more general zeta functions and L-series we wish to consider. Here we recall some of its analytic properties. We begin with the definition of $\Gamma(s)$ as a Mellin transform.

Definition 17.11. The *Mellin transform* of a function $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ is the complex function defined by

$$\mathcal{M}(f)(s) := \int_0^\infty f(t)t^{s-1}dt,$$

whenever this integral converges. It is holomorphic on $\operatorname{Re} s \in (a,b)$ for any interval (a,b) in which the integral $\int_0^\infty |f(t)|t^{\sigma-1}dt$ converges for all $\sigma \in (a,b)$.

Definition 17.12. The *Gamma function*

$$\Gamma(s) := \mathcal{M}(e^{-t})(s) = \int_0^\infty e^{-t}t^{s-1}dt,$$

is the Mellin transform of e^{-t}. Since $\int_0^\infty |e^{-t}|t^{\sigma-1}dt$ converges for all $\sigma > 0$, the integral defines a holomorphic function on $\operatorname{Re}(s) > 0$.

Integration by parts yields

$$\Gamma(s) = \frac{t^s e^{-t}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-t}t^s dt = \frac{\Gamma(s+1)}{s},$$

thus $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 (since $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$), and

$$\Gamma(s+1) = s\Gamma(s) \tag{2}$$

for $\operatorname{Re}(s) > 0$. Equation (2) allows us to extend $\Gamma(s)$ to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, and no other poles.

An immediate consequence of (2) is that for integers $n > 0$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1) = n!,$$

thus the gamma function can be viewed as an extension of the factorial function. The gamma function satisfies many useful identities in addition to (2), including the following.

Theorem 17.13 (Euler's Reflection Formula)**.** *We have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

as meromorphic functions on $\mathbb{C}$ with simple poles at each integer $s \in \mathbb{Z}$.

Proof. Let $f(s) := \Gamma(s)\Gamma(1-s)\sin(\pi s)$. The function $\Gamma(s)\Gamma(1-s)$ has a simple pole at each $s \in \mathbb{Z}$ and no other poles, while $\sin(\pi s)$ has a zero at each $s \in \mathbb{Z}$ and no poles, so $f(s)$ is holomorphic on $\mathbb{C}$. We now note that

$$f(s+1) = \Gamma(s+1)\Gamma(-s)\sin(\pi s + \pi) = -s\Gamma(s)\Gamma(-s)\sin(\pi s) = \Gamma(s)\Gamma(1-s)\sin(\pi s) = f(s),$$

so f is periodic (with period 1). Using the substitution $u = e^t$ we obtain

$$|\Gamma(s)| \le \int_0^\infty |e^{-t} t^{s-1}| dt = \int_{-\infty}^\infty |e^{-e^u} e^{u(s-1)}| e^u du = \int_{-\infty}^\infty e^{u \operatorname{Re}(s) - e^u} du.$$

This implies $|\Gamma(s)|$ is bounded on $\operatorname{Re}(s) \in [1,2]$, hence on $\operatorname{Re}(s) \in [0,1] \cap \operatorname{Im}(s) \ge 1$, via (2). It follows that in the strip $\operatorname{Re}(s) \in [0,1]$ we have

$$|f(s)| = |\Gamma(s)||\Gamma(1-s)||\sin(\pi s)| = O(e^{\operatorname{Im}(s)}),$$

as $\operatorname{Im}(s) \to \infty$, since $|\sin(\pi s)| = \frac{1}{2}|e^{is} - e^{i\bar{s}}| = O(e^{\operatorname{Im}(s)})$. By Lemma 17.14 below, $f(s)$ is constant. To determine the constant, as $s \to 0$ we have $\Gamma(s) \sim \frac{1}{s}$ and $\sin(\pi s) \sim \pi s$, thus

$$f(0) = \lim_{s \to 0} f(s) = \lim_{s \to 0} \Gamma(s)\Gamma(1-s)\sin(\pi s) = \lim_{s \to 0} \frac{1}{s} \cdot 1 \cdot \pi s = \pi,$$

and the theorem follows. ☐

Lemma 17.14. *Let $f(s)$ be a holomorphic function on $\mathbb{C}$ such that $f(s+1) = f(s)$ and $|f(s)| = O(e^{\operatorname{Im}(s)})$ as $\operatorname{Im}(s) \to \infty$ in the vertical strip $\operatorname{Re}(s) \in [0,1]$. Then f is constant.*

Proof. The function

$$g(s) = \frac{f(s) - f(a)}{\sin(\pi(s-a))}$$

is holomorphic on $\mathbb{C}$, since $f(s) - f(a)$ is holomorphic and vanishes at the zeros $a + \mathbb{Z}$ of $\sin(\pi(s-a))$ (all of which are simple). We also have $g(s+1) = g(s)$, and $|g(s)|$ is bounded on $\operatorname{Re}(s) \in [0,1]$, since as $\operatorname{Im}(s) \to \infty$ we have $|f(s) - f(a)| = O(e^{\operatorname{Im}(s)})$ and $|\sin(\pi(s-a))| \sim e^{\pi \operatorname{Im}(s)}$. It follows that $g(s)$ is bounded on $\mathbb{C}$, hence constant, by Liouville's theorem. We must have $g = 0$, since $|g(s)| = O(e^{(1-\pi)\operatorname{Im}(s)}) = o(1)$ as $\operatorname{Im}(s) \to \infty$, and this implies $f(s) = f(a)$ for all $s \in \mathbb{C}$. ☐

Example 17.15. Putting $s = \frac{1}{2}$ in the reflection formula yields $\Gamma(\frac{1}{2})^2 = \pi$, so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Corollary 17.16. *The function $\Gamma(s)$ has no zeros on $\mathbb{C}$.*

Proof. Suppose $\Gamma(s_0) = 0$. The RHS of the reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ is never zero, since $\sin(\pi s)$ has no poles, so $\Gamma(1-s)$ must have a pole at s_0. Therefore $1 - s_0 \in \mathbb{Z}$, equivalently, $s_0 \in \mathbb{Z}$, but for $s_0 \in \mathbb{Z}_{>0}$ we have $\Gamma(s_0) = (s_0 - 1)! \ne 0$, and for $s_0 \in \mathbb{Z}_{\le 0}$ we cannot have $\Gamma(s_0) = 0$ because $\Gamma(s)$ has a pole at all non-positive integers. ☐

17.1.3 Completing the zeta function

Let us now consider the function

$$F(s) := \pi^{-s}\Gamma(s)\zeta(2s),$$

which is holomorphic on $\operatorname{Re}(s) > 1/2$. In the region $\operatorname{Re}(s) > 1/2$ we have an absolutely convergent sum

$$F(s) = \pi^{-s}\Gamma(s)\sum_{n \ge 1} n^{-2s} = \sum_{n \ge 1} (\pi n^2)^{-s}\Gamma(s) = \sum_{n \ge 1} \int_0^\infty (\pi n^2)^{-s} t^{s-1} e^{-t} dt,$$

and the substitution $t = \pi n^2 y$ with $dt = \pi n^2 dy$ yields

$$F(s) = \sum_{n\geq 1}\int_0^\infty (\pi n^2)^{-s}(\pi n^2 y)^{s-1}e^{-\pi n^2 y}\pi n^2 dy = \sum_{n\geq 1}\int_0^\infty y^{s-1}e^{-\pi n^2 y}dy.$$

By the Fubini-Tonelli theorem, we can swap the sum and the integral to obtain

$$F(s) = \int_0^\infty y^{s-1}\sum_{n\geq 1} e^{-\pi n^2 y}dy.$$

We have $\Theta(iy) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 y}$, thus

$$\begin{aligned} F(s) &= \frac{1}{2}\int_0^\infty y^{s-1}(\Theta(iy)-1)dy \\ &= \frac{1}{2}\left(\int_0^1 y^{s-1}\Theta(iy)dy - \frac{1}{s} + \int_1^\infty y^{s-1}(\Theta(iy)-1)dy\right) \end{aligned}$$

We now focus on the first integral on the RHS. The change of variable $t = \frac{1}{y}$ yields

$$\int_0^1 y^{s-1}\Theta(iy)dy = \int_\infty^1 t^{1-s}\Theta(i/t)(-t^{-2})dt = \int_1^\infty t^{-s-1}\Theta(i/t)dt.$$

By Lemma 17.10, $\Theta(i/t) = \sqrt{t}\Theta(it)$, and adding $-\int_1^\infty t^{-s-1/2}dt + \int_1^\infty t^{-s-1/2}dt = 0$ yields

$$\begin{aligned} &= \int_1^\infty t^{-s-1/2}\big(\Theta(it)-1\big)dt + \int_1^\infty t^{-s-1/2}dt \\ &= \int_1^\infty t^{-s-1/2}\big(\Theta(it)-1\big)dt - \frac{1}{1/2 - s}. \end{aligned}$$

Plugging this back into our equation for $F(s)$ we obtain the identity

$$F(s) = \frac{1}{2}\int_1^\infty \big(y^{s-1} + y^{-s-1/2}\big)\big(\Theta(iy)-1\big)dy - \frac{1}{2s} - \frac{1}{1-2s},$$

valid on $\mathrm{Re}(s) > 1/2$. We now observe that $F(s) = F(\frac{1}{2} - s)$ for $s \neq 0, \frac{1}{2}$, which allows us to analytically extend $F(s)$ to a meromorphic function on $\mathbb{C}$ with poles only at $s = 0, 1/2$. Replacing s with $s/2$ leads us to define the *completed zeta function*

$$Z(s) := \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s), \tag{3}$$

which is meromorphic on $\mathbb{C}$ and satisfies the *functional equation*

$$Z(s) = Z(1-s). \tag{4}$$

It has simple poles at 0 and 1 (and no other poles). The only zeros of $Z(s)$ on $\mathrm{Re}(s) > 0$ are the zeros of $\zeta(s)$, since by Corollary 17.16, the gamma function $\Gamma(s)$ has no zeros (and neither does $\pi^{-s/2}$). Thus the zeros of $Z(s)$ on $\mathbb{C}$ all lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

The functional equation also allows us to analytically extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ whose only pole is a simple pole at $s = 1$; the pole of $Z(s)$ at $s = 0$ comes from the pole of $\Gamma(s/2)$ at $s = 0$. The function $\Gamma(s/2)$ also has poles at $-2, -4, \ldots$ where $Z(s)$ does not, so our extended $\zeta(s)$ must have zeros at $-2, -4, \ldots$. These are *trivial zeros*;

all the interesting zeros of $\zeta(s)$ lie in the critical strip and are conjectured to lie only on the critical line $\mathrm{Re}(s) = 1/2$ (this is the Riemann hypothesis).

We can compute $\zeta(0)$ using the functional equation. From (3) and (4) we have

$$\zeta(s) = \frac{Z(s)}{\pi^{-s/2}\Gamma(\frac{s}{2})} = \frac{Z(1-s)}{\pi^{-s/2}\Gamma(\frac{s}{2})} = \frac{\pi^{\frac{(s-1)}{2}}\Gamma(\frac{1-s}{2})}{\pi^{-s/2}\Gamma(\frac{s}{2})}\zeta(1-s) = \frac{\pi^{s-1/2}\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s). \tag{5}$$

We know that $\zeta(s)$ has a simple pole with residue 1 at $s = 1$, so

$$1 = \lim_{s\to 1^+}(s-1)\zeta(s) = \lim_{s\to 1^+}\frac{(s-1)\pi^{s-1/2}\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s).$$

When $s = 1$, the denominator on the RHS is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which cancels $\pi^{1-1/2} = \sqrt{\pi}$ in the numerator. Using $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$ to shift $\Gamma(\frac{1-s}{2})$ in the numerator yields

$$1 = \lim_{s\to 1^+}(s-1)\tfrac{2}{1-s}\Gamma\left(\tfrac{3-s}{2}\right)\zeta(1-s) = -2\Gamma(1)\zeta(0) = -2\zeta(0).$$

Thus $\zeta(0) = -1/2$.

Using the reflection formula to replace $\Gamma(\frac{s}{2}) = \pi/(\Gamma(\frac{2-s}{2})\sin(\frac{\pi s}{2}))$ in (5), we have

$$\zeta(s) = \pi^{s-3/2}\Gamma(\tfrac{1-s}{2})\Gamma(\tfrac{2-s}{2})\sin(\tfrac{\pi s}{2})\zeta(1-s).$$

Applying the duplication formula $\Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$ with $z = \frac{1-s}{2}$ then yields

$$\zeta(s) = 2^s\pi^{s-1}\sin(\tfrac{\pi s}{2})\Gamma(1-s)\zeta(1-s), \tag{6}$$

which is how one often sees the functional equation for $\zeta(s)$ written.

17.2 Gamma factors and a holomorphic zeta function

If we write out the Euler product for the completed zeta function, we have

$$Z(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2}) \cdot \prod_p (1-p^{-s})^{-1}.$$

One should think of this as a product over the places of the field $\mathbb{Q}$; the leading factor

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(\tfrac{s}{2})$$

that distinguishes the completed zeta function $Z(s)$ from $\zeta(s)$ corresponds to the real archimedean place of $\mathbb{Q}$. When we discuss Dedekind zeta functions in a later lecture we will see that there are gamma factors $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ associated to each of the real and complex places of a number field.

If we insert an additional factor of $\binom{s}{2} := \frac{s(s-1)}{2}$ in $Z(s)$ we can remove the poles at 0 and 1, yielding a function $\xi(s)$ holomorphic on $\mathbb{C}$. This yields Riemann's seminal result.

Theorem 17.17 (Analytic Continuation II)**.** *The function*

$$\xi(s) := \tbinom{s}{2}\Gamma_{\mathbb{R}}(s)\zeta(s)$$

is holomorphic on $\mathbb{C}$ and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

The zeros of $\xi(s)$ all lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

Remark 17.18. We will usually work with $Z(s)$ and deal with the poles rather than making it holomorphic by introducing additional factors; some authors use $\xi(s)$ to denote our $Z(s)$.

17.3 Zeros in the critical strip

The zeros of $\xi(s)$ in the critical strip are precisely the zeros of $\zeta(s)$ in the critical strip. Let $N(T)$ denote the number of zeros of $\xi(s)$ in the critical strip that satisfy $0 < \operatorname{im} s < T$. If we fix $\epsilon > 0$ and let R be the rectangle $\{-\epsilon \le \operatorname{Re}(s) \le 1+\epsilon,\ 0 \le \operatorname{im} s \le T\}$, we can compute $N(T)$ using Cauchy's argument principle via

$$N(T) = \frac{1}{2\pi i}\int_{\partial R}\frac{\xi'(s)}{\xi(s)}ds,$$

provided that there are no zeros on the lines $\operatorname{im} s = 0$ and $\operatorname{im} s = T$. From this formula and the functional equation one derive the asymptotic formula

$$N(T) \sim \frac{1}{2\pi}T\log\left(\frac{T}{2\pi e}\right),$$

along with an explicit error term that allows one to compute the integer $N(T)$ exactly. Note that this formula implies that there are infinitely many zeros in the critical strip. The Riemann hypothesis states that all of these zeros lie on the critical line $\operatorname{im} s = 1/2$.

One can count zeros on the critical line by counting zeros of the *Hardy Z-function*

$$e^{i\theta(t)}\zeta(1/2 + it)$$

in a region $0 \le t \le T$, where $\theta(t)$ is the *Riemann-Siegel function*

$$\theta(t) \coloneqq \arg\left(\Gamma\left(\frac{2it+1}{4}\right)\right) - \frac{\log\pi}{2}t.$$

There are asymptotic expansions of the Hardy Z-function that allow one to do this efficiently (one counts sign changes and checks for multiple roots). By comparing the result to $N(T)$ one can determine whether all the zeros in the critical strip with $0 < \operatorname{im} s < T$ lie on the critical line or not. This has been done for values of T exceeding 10^{13}; more precisely, it has been verified that when ordered by their imaginary parts, the first 10^{13} zeros above the real axis all lie on the critical line; see [2] for details.

References

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[2] Xavier Gourdon, *The 10^{13} first zeros of the Riemann zeta function and zeros computation at very large height*, preprint, 2004.