13 Global fields and the product formula

Up to this point we have defined global fields as finite extensions of $\mathbb{Q}$ (number fields) or of $\mathbb{F}_q(t)$ (global function fields). Our goal in this lecture is to prove a generalization of the product formula that you proved on Problem Set 1 for $K = \mathbb{Q}$ and $K = \mathbb{F}_q(t)$, which will then allow us to give a more natural definition of global fields: they are fields whose completions are local fields and which satisfy a suitable product formula.

13.1 Places of a field

Definition 13.1. Let K be a field. A *place* of K is an equivalence class of nontrivial absolute values on K. Recall that the completion of K at an absolute value depends only on its equivalence class, so there is a one-to-one correspondence between places of K and completions of K. We may use M_K to denote the set of places of K, and for each place v we use $|\ |_v$ to denote any representative absolute and K_v to denote the completion of K with respect to $|\ |_v$ (this does not depend on the choice of $|\ |_v$). We call a place v *archimedean* when K_v is archimedean and *nonarchimedean* otherwise.

Now let K be a global field. By Corollary 9.7, for any place v of K the completion K_v is a local field. From our classification of local fields (Theorem 9.9), if K_v is archimedean then $K_v \simeq \mathbb{R}$ or $K_v \simeq \mathbb{C}$, and otherwise the absolute value of K_v is induced by a discrete valuation that we also denote v; note that while the absolute value $|x|_v := c^{-v(x)}$ depends on a choice of $c \in [0,1]$, the discrete valuation $v\colon K_v \to \mathbb{Z}$ is uniquely determined. We now introduce the following terminology:

- if $K_v \simeq \mathbb{R}$ then v is a *real place*;
- if $K_v \simeq \mathbb{C}$ then v is a *complex place*;
- if $|\ |_v$ is induced by a discrete valuation $v_\mathfrak{p}$ corresponding to a prime $\mathfrak{p}$ of K then v is a *finite place*; otherwise v is an *infinite place*.

Every finite place is nonarchimedean. Infinite places are archimedean in characteristic zero and nonarchimedean otherwise. Every archimedean place is an infinite place, but nonarchimedean places may be finite or infinite (the latter only in positive characteristic).

Example 13.2. As you proved on Problem Set 1, the set $M_\mathbb{Q}$ consists of finite places p corresponding to p-adic absolute values $|\ |_p$, and a single archimedean infinite place ∞ corresponding to the Euclidean absolute value $|\ |_\infty$. The set $M_{\mathbb{F}_q(t)}$ consist of finite places corresponding to irreducible polynomials in $\mathbb{F}_q[t]$ and a single nonarchimedean infinite place ∞ corresponding to the absolute value $|\cdot|_\infty := q^{\deg(\cdot)}$.

Remark 13.3. There is nothing special about the infinite place of $\mathbb{F}_q(t)$, it is an artifact of our choice of the separating element t, which we could change by applying an automorphism $t \mapsto (at+b)/(ct+d)$ of $\mathbb{F}_q(t)$. If we put $z := 1/t$ and consider $\mathbb{F}_q(z) \simeq \mathbb{F}_q(t)$, the absolute value $|\ |_\infty$ on $\mathbb{F}_q(t)$ is the same as the absolute value $|\ |_z$ on $\mathbb{F}_q(z)$ corresponding to the irreducible polynomial $z \in \mathbb{F}_q[z]$. This is analogous to the situation with the projective line $\mathbb{P}^1$, where we may distinguish the projective point $(1:0)$ as the "point at infinity", but this distinction is not invariant under automorphisms of $\mathbb{P}^1$.

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Definition 13.4. If L/K is an extension of global fields, for every place w of L, any absolute value $|\ |_w$ that represents the equivalence class w restricts to an absolute value on K that represents a place v of K that is independent of the choice of $|\ |_w$. We write $w|v$ to indicate this relationship and say that w *extends* v or that w *lies above* v.

Theorem 13.5. *Let L/K be a finite separable extension of global fields and let v be a place of K. We have an isomorphism of finite étale K_v-algebras*

$$L \otimes_K K_v \xrightarrow{\sim} \prod_{w|v} L_w$$

defined by $\ell \otimes x \mapsto (\ell x, \ldots, \ell x)$.

For nonarchimedean places this follows from part (v) of Theorem 11.23, but here we give a different proof that works for any place of K.

Proof. The separable extension L/K is a finite étale K-algebra, so the base change $L \otimes_K K_v$ is a finite étale K_v-algebra, by Proposition 4.36, and is therefore isomorphic to a finite product $\prod_{i \in I} L_i$ of finite separable extensions L_i of K_v, each of which is a local field (any finite extension of a local field is a local field). We just need to show that there is a one-to-one correspondence between the sets of local fields $\{L_i : i \in I\}$ and $\{L_w : w|v\}$.

Let us fix an absolute value $|\ |_v$ on K_v representing the place v. Each L_i is a local field extending K_v, and therefore has a unique absolute value $|\ |_w$ that restricts to $|\ |_v$; this follows from Theorem 10.6 when v is nonarchimedean and is obvious when v is archimedean, since then either $K_v \simeq L_w$ or $K_v \simeq \mathbb{R} \subseteq \mathbb{C} \simeq L_w$ and the Euclidean absolute value on $\mathbb{R}$ is the restriction of the Euclidean absolute value on $\mathbb{C}$. The map $L \hookrightarrow L \otimes_K K_v \simeq \prod_i L_i \twoheadrightarrow L_i$ allows us to view L as a subfield of each L_i, so the absolute value $|\ |_w$ on L_i restricts to an absolute value on L that uniquely determines a place $w|v$. This defines a map $\phi\colon \{L_i : i \in I\} \to \{L_w : w|v\}$ that we will show is a bijection satisfying $\phi(L_i) \simeq L_i$.

We may view $L \otimes_K K_v \simeq \prod_i L_i$ as an isomorphism of topological rings, since both sides are finite dimensional vector spaces over the complete field K_v and thus have a unique topology induced by the sup norm, by Proposition 10.5, and this topology agrees with the product topology on $\prod_i L_i$. The image of the canonical embedding $L \hookrightarrow L \otimes_K K_v$ defined by $\ell \mapsto \ell \otimes 1$ is dense because $K \subseteq L$ is dense in K_v: given any $\ell \otimes x$ in $L \otimes_K K_v$ with $\ell \in L$ and $x \in K_v$, we can choose $y \in K^\times$ arbitrarily close to x so that $\ell y \otimes 1 = \ell \otimes y$ is an element of the image of L arbitrarily close to $\ell \otimes x$ (and similarly for sums of pure tensors). The image of L is therefore also dense in $\prod_i L_i$ and has dense image under the projections $\prod_i L_i \twoheadrightarrow L_i$ and $\prod_i L_i \twoheadrightarrow L_i \times L_j$ $(i \neq j)$.

If $\phi(L_i) = L_w$ then $L_i \simeq L_w$ since L is dense in the complete field L_i, and L_w is the completion of L with respect to the restriction of the absolute value on L_i to L, by the universal property of completions (Proposition 8.4). To show that ϕ is injective, note that if $\phi(L_i) = \phi(L_j) = L_w$ for some $i \neq j$ we obtain a contradiction because the image of the diagonal embedding $L \to L_w \times L_w$ is not dense in $L_w \times L_w$ (its closure is isomorphic to L_w), but the image of L is dense in $L_i \times L_j$.

It remains only to show that ϕ is surjective. For each $w|v$ we may define a continuous homomorphism of finite étale K_v-algebras and topological rings:

$$\begin{aligned} \varphi_w\colon L \otimes_K K_v &\to L_w \\ \ell \otimes x &\mapsto \ell x. \end{aligned}$$

The map φ_w is surjective because its image contains L and is complete, and L_w is the completion of L. It follows from Corollary 4.32 that φ_w factors through the projection of $L \otimes_K K_v \simeq \prod_i L_i$ on to one of its factors L_i and induces a homeomorphism from L_i to L_w. It follows that $L_i \simeq L_w$ as topological fields, so $\phi(L_i) = L_w$ and ϕ is surjective. $\square$

Corollary 13.6. *Let L/K be a finite separable extension of global fields, v be a place of K, and $f \in K[x]$ be an irreducible polynomial such that $L \simeq K[x]/(f(x))$. There is a one-to-one correspondence between the irreducible factors of f in $K_v[x]$ and the places of L lying above v. If $f = f_1 \cdots f_r$ is the factorization of f in $K_v[x]$, then we can order the set $\{w|v\} = \{w_1, \ldots, w_r\}$ so that $L_{w_i} \simeq K_v[x]/(f_i(x))$ for $1 \le i \le r$.*

Proof. Note that the f_i are distinct because f is separable over K and therefore separable over every extension of K, including K_v. The corollary then follows from Proposition 4.33, Corollary 4.39, and Theorem 13.5. $\square$

Given a finite separable extension of global fields L/K and a place v of K, if we fix an algebraic closure $\overline{K}_v$ of K_v and consider the set $\operatorname{Hom}_K(L, \overline{K}_v)$ of K-embeddings of L into $\overline{K}_v$, the Galois group $\operatorname{Gal}(\overline{K}_v/K_v)$ acts on the set $\operatorname{Hom}_K(L, \overline{K}_v)$ via composition: given $\sigma \in \operatorname{Gal}(\overline{K}_v/K_v)$ and $\tau \in \operatorname{Hom}_K(L, \overline{K}_v)$, we have $\sigma \circ \tau \in \operatorname{Hom}_K(L, \overline{K}_v)$, and this clearly defines a group action (composition is associative and the identity acts trivially).

Corollary 13.7. *Let L/K be a finite separable extension of global fields and v a place of K. We have a bijection*

$$\operatorname{Hom}_K(L, \overline{K}_v)/\operatorname{Gal}(\overline{K}_v/K_v) \longleftrightarrow \{w|v\},$$

between $\operatorname{Gal}(\overline{K}_v/K_v)$-orbits of K-embeddings of L into $\overline{K}_v$ and the places of L above v.

Proof. By the primitive element theorem, we may assume $L \simeq K(\alpha) = K[x]/(f)$ for some $\alpha \in L$ with minimal polynomial $f \in K[x]$. We then have a bijection between $\operatorname{Hom}_K(L, \overline{K}_v)$ and the roots α_i of f in $\overline{K}_v$ that is compatible with the action of $\operatorname{Gal}(\overline{K}_v/K_v)$ on both sets. If $f = f_1 \cdots f_r$ is the factorization of f in $K_v[x]$, each f_i corresponds to an orbit of the action of $\operatorname{Gal}(\overline{K}_v/K_v)$ on the roots of f, and by the previous corollary, these are in one-to-one correspondence with the places of L above v. $\square$

For $K = \mathbb{Q}$ and $v = \infty$, Corollary 13.7 implies that $\operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})/\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ is in bijection with the set $\{w|\infty\}$ of infinite places of the number field L; note that $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ is the cyclic group of order 2 generated by complex conjugation, so the orbits of $\operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})$ all have size 1 or 2, depending on whether the embedding of L into $\mathbb{C}$ is fixed by complex conjugation or not. Each real place w corresponds to a $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$-orbit of size 1; this occurs for the elements of $\operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})$ whose image lies in $\mathbb{R}$ and may also be viewed as elements of $\operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{R})$. Each complex place corresponds to a $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$-orbit of size two in $\operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})$; these are conjugate pairs whose images do not lie in $\mathbb{R}$.

Definition 13.8. Let K be a number field. Elements of $\operatorname{Hom}_{\mathbb{Q}}(K, \mathbb{R})$ are *real embeddings*, and elements of $\operatorname{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ whose image does not lie in $\mathbb{R}$ are *complex embeddings*.

There is a one-to-one correspondence between real embeddings and real places, but complex embeddings come in conjugate pairs; each pair of complex embeddings corresponds to a single complex place.

Corollary 13.9. *Let K be a number field with r real places and s complex places. Then*

$$[K : \mathbb{Q}] = r + 2s.$$

Proof. We may write $K \simeq \mathbb{Q}[x]/(f)$ for some irreducible separable $f \in \mathbb{Q}[x]$, and we then have $[K:\mathbb{Q}] = \deg f = \#\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C})$, since there is a one-to-one correspondence between $\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ and the roots of f. The action of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on $\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ has r orbits of size 1, and s orbits of size 2, and the corollary follows. □

Example 13.10. Let $K = \mathbb{Q}[x]/(x^3-2)$. There are three embeddings $K \hookrightarrow \mathbb{C}$, one for each root of x^3-2; explicitly:

$$(1)\ x \mapsto \sqrt[3]{2}, \qquad (2)\ x \mapsto e^{2\pi i/3}\cdot\sqrt[3]{2}, \qquad (3)\ x \mapsto e^{4\pi i/3}\cdot\sqrt[3]{2}.$$

The first embedding is real, while the second two are complex and conjugate to each other. Thus K has $r=1$ real place and $s=1$ complex place, and we have $[K:\mathbb{Q}] = 1\cdot 1 + 2\cdot 1 = 3$.

We conclude this section with a result originally due to Brill [2] that relates the parity of the number of complex places to the sign of the absolute discriminant of a number field.

Proposition 13.11. *Let K be a number field with s complex places. The sign of the absolute discriminant $D_K \in \mathbb{Z}$ is $(-1)^s$.*

Proof. Let $\alpha_1,\ldots,\alpha_n$ be a $\mathbb{Z}$-basis for $\mathcal{O}_K$, let $\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C}) = \{\sigma_1,\ldots,\sigma_n\}$ and consider the matrix $A := [\sigma_i(\alpha_j)]_{ij}$ with determinant $\det A =: x + yi \in \mathbb{C}$; recall that $D_K = (\det A)^2$, by Proposition 12.6. Each real embedding σ_i corresponds to a row of A fixed by complex conjugation, while each pair of complex conjugate embeddings $\sigma_i, \bar\sigma_i$ corresponds to a pair of rows of A that are interchanged by complex conjugation. Swapping a pair of rows negates the determinant, thus $\det \bar A = (-1)^s \det A$, and we have

$$x + yi = \det A = (-1)^s \det \bar A = (-1)^s(x - yi).$$

Either $(-1)^s = 1$, in which case $y = 0$ and $D_K = x^2$ has sign $+1 = (-1)^s$, or $(-1)^s = -1$, in which case $x = 0$ and $D_K = -y^2$ has sign $-1 = (-1)^s$. □

13.2 Haar measures

Definition 13.12. Let X be a locally compact Hausdorff space. The *σ-algebra* Σ of X is the collection of subsets of X generated by the open and closed sets under countable unions and countable intersections. Its elements are *Borel sets*, or *measurable sets*. A *Borel measure* on X is a countably additive function

$$\mu\colon \Sigma \to \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

A *Radon measure* on X is a Borel measure on X that additionally satisfies

1. $\mu(S) < \infty$ if S is compact,
2. $\mu(S) = \inf\{\mu(U) : S \subseteq U,\ U \text{ open}\}$,
3. $\mu(S) = \sup\{\mu(C) : C \subseteq S,\ C \text{ compact}\}$,

for all Borel sets S.[1]

[1]Some authors additionally require X to be σ-compact (a countable union of compact sets). Local fields are σ-compact so this distinction will not concern us.

Definition 13.13. A *locally compact group* is a topological group that is Hausdorff and locally compact.[2] A (*left*) *Haar measure* μ on a locally compact group G is a nonzero Radon measure that is *translation invariant*, meaning that

$$\mu(S) = \mu(x + S)$$

for all $x \in G$ and measurable $S \subseteq X$ (we write the group operation additively because we have in mind the additive group of a local field). A compact group is a locally compact group that is compact; in compact groups every measurable set has finite measure.

One defines a right Haar measure analogously, but in most cases they coincide and in our situation we are working with an abelian group (the additive group of a field), in which case they necessarily do. The key fact we need about Haar measures is that they exist and are unique up to scaling. For compact groups existence was proved by Haar and uniqueness by von Neumann; the general result for locally compact groups was proved by Weil.

Theorem 13.14 (Weil)**.** *Every locally compact group G has a Haar measure. If μ and μ' are two Haar measure on G, then there is a positive real number λ such that $\mu'(S) = \lambda\mu(S)$ for all measurable sets S.*

Proof. See [3, §7.2]. $\square$

Example 13.15. The standard Lebesgue measure μ on $\mathbb{R}^n$ with $\mu(\prod_i [a_i, b_i]) = \prod_i |b_i - a_i|$ is the unique Haar measure on $\mathbb{R}^n$ for which the unit cube has measure 1.[3]

The additive group of a local field K is a locally compact group (it is a metric space, hence Hausdorff). For compact groups G, it is standard to normalize the Haar measure so that $\mu(G) = 1$, but local fields are never compact, and we will always have $\mu(K) = \infty$. For nonarchimedean local fields the valuation ring $A = B_{\leq 1}(0)$ is a compact group, and it is then natural to normalize the Haar measure on K so that $\mu(A) = 1$. The key point is that there is a unique absolute value on K that is compatible with every Haar measure μ on K, no matter how it is normalized.

Proposition 13.16. *Let K be a local field with discrete valuation v, residue field k, and absolute value*

$$|\cdot|_v := (\#k)^{-v(\cdot)},$$

and let μ be a Haar measure on K. For every $x \in K$ and measurable set $S \subseteq K$ we have

$$\mu(xS) = |x|_v \mu(S).$$

Moreover, the absolute value $|\ |_v$ is the unique absolute value compatible with the topology on K for which this is true.

Proof. Let A be the valuation ring of K with maximal ideal $\mathfrak{p}$. The proposition clearly holds for $x = 0$, so let $x \neq 0$. The map $\phi_x \colon y \mapsto xy$ is an automorphism of the additive group of K, and it follows that the composition $\mu_x = \mu \circ \phi_x$ is a Haar measure on K, hence

[2] Note that the Hausdorff assumption is part of the definition. Some authors include it in the definition of locally compact, and some also include it in the definition of compact (Bourbaki, for example); the convention varies by field and with geography. But everyone agrees that locally compact groups are Hausdorff.

[3] Strictly speaking, the Haar measure on $\mathbb{R}^n$ is the restriction of the Lebesgue measure to the σ-algebra.

a multiple of μ, say $\mu_x = \lambda_x \mu$, for some $\lambda_x \in \mathbb{R}_{>0}$. Define the function $\chi\colon K^\times \to \mathbb{R}_{>0}$ by $\chi(x) := \lambda_x = \mu_x(A)/\mu(A)$. Then $\mu_x = \chi(x)\mu$, and for all $x, y \in K^\times$ we have

$$\chi(xy) = \frac{\mu_{xy}(A)}{\mu(A)} = \frac{\mu_x(yA)}{\mu(A)} = \frac{\chi(x)\mu_y(A)}{\mu(A)} = \frac{\chi(x)\chi(y)\mu(A)}{\mu(A)} = \chi(x)\chi(y).$$

Thus χ is multiplicative, and we claim that in fact $\chi(x) = |x|_v$ for all $x \in K^\times$. Since both χ and $|\cdot|_v$ are multiplicative, it suffices to consider $x \in A - \{0\}$. For any such x, the ideal xA is equal to $\mathfrak{p}^{v(x)}$, since A is a DVR. The residue field $k := A/\mathfrak{p}$ is finite, hence A/xA is also finite; indeed it is a k-vector space of dimension $v(x)$ and has cardinality $[A : xA] = (\#k)^{v(x)}$. Writing A as a finite disjoint union of cosets of xA, we have

$$\mu(A) = [A : xA]\mu(xA) = (\#k)^{v(x)}\chi(x)\mu(A),$$

and therefore $\chi(x) = (\#k)^{-v(x)} = |x|_v$ as claimed. It follows that

$$\mu(xS) = \mu_x(S) = \chi(x)\mu(S) = |x|_v\mu(S),$$

for all $x \in K$ and measurable $S \subseteq K$. To prove uniqueness, if $|\ |$ is an absolute value on K that induces the same topology as $|\ |_v$ then for some $0 < c < 1$ we have $|x| = |x|_v^c$ for all $x \in K^\times$. Let us fix $x \in K^\times$ with $|x|_v \neq 1$ (take any x with $v(x) \neq 0$). If $|\ |$ also satisfies $\mu(xS) = |x|\mu(S)$ then

$$\frac{\mu(xA)}{\mu(A)} = |x| = |x|_v^c = \left(\frac{\mu(xA)}{\mu(A)}\right)^c,$$

which implies $c = 1$, meaning that $|\ |$ and $|\ |_v$ are the same absolute value. □

13.3 The product formula for global fields

Definition 13.17. Let K be a global field. For each place v of K the *normalized absolute value* $\|\ \|_v\colon K_v \to \mathbb{R}_{\geq 0}$ on the completion of K at v is defined by

$$\|x\|_v := \frac{\mu(xS)}{\mu(S)},$$

where μ is a Haar measure on K_v and $S \subseteq K_v$ is a measurable set with finite nonzero measure (such as the set $\{x \in K_v : |x|_v \leq 1\}$, for example).

This definition is independent of the choice of μ and S (by Theorem 13.14). If v is nonarchimedean then the normalized absolute value $\|\ \|_v$ is precisely the absolute value $|\ |_v$ defined in Proposition 13.16. If v is a real place then the normalized absolute value $\|\ \|_v$ is just the usual Euclidean absolute value $|\ |_\mathbb{R}$ on $\mathbb{R}$, since for the Euclidean Haar measure $\mu_\mathbb{R}$ on $\mathbb{R}$ we have $\mu_\mathbb{R}(xS) = |x|_\mathbb{R}\mu_\mathbb{R}(S)$ for every measurable set S. But when v is a complex place the normalized absolute value $\|\ \|_v$ is the **square** of the Euclidean absolute value $|\ |_\mathbb{C}$ on $\mathbb{C}$, since in $\mathbb{C}$ we have $\mu_\mathbb{C}(xS) = |x|_\mathbb{C}^2\mu_\mathbb{C}(S)$.

Remark 13.18. When v is a complex place the normalized absolute value $\|\ \|_v$ is **not** an absolute value, because it does not satisfy the triangle inequality. For example, if $K = \mathbb{Q}(i)$ and $v|\infty$ is the complex place of K then $\|1\|_v = |1|_\mathbb{C}^2 = 1$ but

$$\|1+1\|_v = \|2\|_v = |2|_\mathbb{C}^2 = 4 > 2 = \|1\|_v + \|1\|_v.$$

Nevertheless, the normalized absolute value $\|\ \|_v$ is always multiplicative and compatible with the topology on K_v in the sense that the open balls $B_{<r}(x) := \{y \in K_v : \|y - x\|_v < r\}$ are a basis for the topology on K_v; these are the properties that we care about for the product formula (and for the topology on the ring of adéles $\mathbb{A}_K$ that we will see later).

Lemma 13.19. *Let L/K be a finite separable extension of global fields, let v be a place of K and let $w|v$ be a place of L. Then*

$$\|x\|_w = \|\mathrm{N}_{L_w/K_v}(x)\|_v.$$

Proof. The lemma is trivially true if $[L_w : K_v] = 1$ so assume $[L_w : K_v] > 1$. If v is archimedean then $L_w \simeq \mathbb{C}$ and $K_v \simeq \mathbb{R}$, in which case for any $x \in L_w$ we have

$$\|x\|_w = \mu(xS)/\mu(S) = |x|_{\mathbb{C}}^2 = |x\overline{x}|_{\mathbb{R}} = |\mathrm{N}_{\mathbb{C}/\mathbb{R}}(x)|_{\mathbb{R}} = \|\mathrm{N}_{L_w/K_v}(x)\|_v,$$

where $|\ |_{\mathbb{R}}$ and $|\ |_{\mathbb{C}}$ are the Euclidean absolute values on $\mathbb{R}$ and $\mathbb{C}$.

We now assume v is nonarchimedean. Let π_v and π_w be uniformizers for the local fields K_v and L_w, respectively, and let f be the degree of the corresponding residue field extension k_w/k_v. Without loss of generality, we may assume $x = \pi_w^{w(x)}$, since $\|x\|_v = |x|_v$ depends only on $w(x)$. Theorem 6.10 and Proposition 13.16 imply

$$\|\mathrm{N}_{L_w/K_v}(\pi_w)\|_v = \|\pi_v^f\|_v = (\#k_v)^{-f},$$

so $\|\mathrm{N}_{L_w/K_v}(x)\|_v = (\#k_v)^{-fw(x)}$. Proposition 13.16 then implies

$$\|x\|_w = (\#k_w)^{-w(x)} = (\#k_v)^{-fw(x)} = \|\mathrm{N}_{L_w/K_v}(x)\|_v. \qquad \square$$

Remark 13.20. Note that if v is a nonarchimedean place of K extended by a place $w|v$ of L/K, the absolute value $\|\ \|_w$ is **not** the unique absolute value on L_w that extends the absolute value on $\|\ \|_v$ on K_v given by Theorem 10.6, it differs by a power of $n = [L_w : K_v]$, but it is equivalent to it. It might seem strange to use a normalization here that does not agree with the one we used when considering extensions of local fields in Lecture 9. The difference is that here we are thinking about a single global field K that has many different completions K_v, and we want the normalized absolute values on the various K_v to be compatible (so that the product formula will hold). By contrast, in Lecture 9 we considered various extensions L_w of a single local field K_v and wanted to normalize the absolute values on the L_w compatibly so that we could work in K_v and any of its extensions (all the way up to $\overline{K}_v$) using the same absolute value. These two objectives cannot be met simultaneously and it is better to use the "right" normalization in each setting.

Theorem 13.21 (Product Formula)**.** *Let L be a global field. For all $x \in L^\times$ we have*

$$\prod_{v \in M_L} \|x\|_v = 1,$$

where $\|\ \|_v$ denotes the normalized absolute value for each place $v \in M_L$.

Proof. The global field L is a finite separable extension of $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$.[4] Let p be a place of K. By Theorem 13.5, any basis for L as a K-vector space is also a basis for

$$L \otimes_K K_p \simeq \prod_{v|p} L_v$$

[4]Here we are using the fact that if $\mathbb{F}_q$ is the field of constants of L (the largest finite field in L), then L is a finite extension of $\mathbb{F}_q(z)$ and we can choose some $t \in \mathbb{F}_q(z) - \mathbb{F}_q$ so that $\mathbb{F}_q(z) \simeq \mathbb{F}_q(t)$ and $L/\mathbb{F}_q(t)$ is separable (such a t is called a *separating element*).

as a K_v-vector space. Thus

$$\mathrm{N}_{L/K}(x) = \mathrm{N}_{(L\otimes_K K_p)/K_p}(x) = \prod_{v|p} \mathrm{N}_{L_v/K_p}(x).$$

Taking normalized absolute values on both sides yields

$$\left\|\mathrm{N}_{L/K}(x)\right\|_p = \prod_{v|p} \|\mathrm{N}_{L_v/K_p}(x)\|_p = \prod_{v|p} \|x\|_v.$$

We now take the product of both sides over all places $p \in M_K$ to obtain

$$\prod_{p\in M_K} \|\mathrm{N}_{L/K}(x)\|_p = \prod_{p\in M_K}\prod_{v|p} \|x\|_v = \prod_{v\in M_L} \|x\|_v.$$

The LHS is equal to 1, by the product formula for K proved on Problem Set 1. □

With the product formula in hand, we can now give an axiomatic definition of a global field, which up to now we have simply defined as a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$, due to Emil Artin and George Whaples [1].

Definition 13.22. A *global field* is a field K with at least one place whose completion at each of its places $v \in M_K$ is a local field K_v, and which has a product formula of the form

$$\prod_{v\in M_K} \|x\|_v = 1,$$

where each normalized absolute value $\|\ \|_v \colon K_v \to \mathbb{R}_{\geq 0}$ satisfies $\|\ \|_v = |\ |_v^{m_v}$ for some absolute value $|\ |_v$ representing v and some fixed $m_v \in \mathbb{R}_{>0}$.

Theorem 13.23 (Artin-Whaples)**.** *Every global field is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$.*

Proof. See Problem Set 7. □

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