14 The Minkowski bound and finiteness results

14.1 Lattices in real vector spaces

Recall that for an integral domain A with fraction field K, an A-*lattice* in a finite dimensional K-vector space V is a finitely generated A-submodule of V that contains a K-basis for V (see Definition 5.9). We now want to specialize to the case $A = \mathbb{Z}$, but rather than working with the fraction field $K = \mathbb{Q}$ we will instead work with its archimedean completion $\mathbb{R}$. Note that a finitely generated $\mathbb{Z}$-submodule of a vector space is necessarily a free module, since $\mathbb{Z}$ is a PID and a submodule of a vector space must be torsion-free. Now V is an $\mathbb{R}$-vector space of some finite dimension n, and has a canonical structure as a topological metric space isomorphic to $\mathbb{R}^n$ (by Proposition 10.5, there is a unique topology on V compatible with the topology of $\mathbb{R}$, because $\mathbb{R}$ is complete). This topology makes V a locally compact Hausdorff space, thus V is a locally compact group and is thus equipped with a Haar measure μ that is unique up to scaling, by Theorem 13.14.

Definition 14.1. A subgroup H of a topological group G is *discrete* if the subspace topology on H is the discrete topology (every point is open), and *cocompact* if H is a normal subgroup of G and the quotient G/H is compact (here G/H denotes the group G/H with the quotient topology given by identifying elements of G that lie in the same coset of H).

Definition 14.2. Let V be an $\mathbb{R}$-vector space of finite dimension. A (full) *lattice* in V is a $\mathbb{Z}$-submodule generated by an $\mathbb{R}$-basis for V; equivalently, a discrete cocompact subgroup.

See Problem Set 7 for a proof that these two definitions are equivalent.

Remark 14.3. A discrete subgroup of a Hausdorff topological group is always closed; see [1, III.2.1.5] for a proof. This implies that the quotient of a Hausdorff topological group by a normal discrete subgroup is Hausdorff (which is false for topological spaces in general); see [1, III.2.1.18]. It follows that the quotient of a Hausdorff topological group (including all locally compact groups) by a discrete cocompact subgroup is a compact group. These facts are easy to see in the case of lattices: $\mathbb{Z}$ is closed in $\mathbb{R}$ (as the complement of a union of open intervals), so $\mathbb{Z}^n$ is closed in $\mathbb{R}^n$. Given a lattice Λ in V, each $\mathbb{Z}$-basis for Λ determines an isomorphism of topological groups $\Lambda \simeq \mathbb{Z}^n$ and $V \simeq \mathbb{R}^n$, and the quotient $V/\Lambda \simeq \mathbb{R}^n/\mathbb{Z}^n \simeq (\mathbb{R}/\mathbb{Z})^n$ (an n-*torus*), is compact Hausdorff and thus a compact group.

Remark 14.4. You might ask why we are using the archimedean completion $\mathbb{R} = \mathbb{Q}_\infty$ rather than some other completion $\mathbb{Q}_p$. The reason is that $\mathbb{Z}$ is not a discrete subgroup of $\mathbb{Q}_p$ (elements of $\mathbb{Z}$ can be arbitrarily close to 0 under the p-adic metric).

Any basis $v_1, \ldots, v_n$ for V determines a parallelepiped

$$F(v_1, \ldots, v_n) := \{t_1v_1 + \cdots + t_nv_n : t_1, \ldots, t_n \in [0, 1)\}$$

that we may view as the unit cube by fixing an isomorphism $\varphi\colon V \xrightarrow{\sim} \mathbb{R}^n$ that maps $(v_1, \ldots, v_n)$ to the standard basis of unit vectors for $\mathbb{R}^n$. It then makes sense to normalize the Haar measure μ so that $\mu(F(v_1, \ldots, v_n)) = 1$, and we then have $\mu(S) = \mu_{\mathbb{R}^n}(\varphi(S))$ for every measurable set $S \subseteq V$, where $\mu_{\mathbb{R}^n}$ denotes the standard Lebesgue measure on $\mathbb{R}^n$.

For any other basis $e_1, \ldots, e_n$ of V, if we let $E = [e_{ij}]_{ij}$ be the matrix whose jth column expresses $e_j = \sum_i e_{ij}v_i$, in terms of our normalized basis $v_1, \ldots, v_n$, then

$$\mu(F(e_1, \ldots, e_n)) = |\det E| = \sqrt{\det E^t \det E} = \sqrt{\det(E^tE)} = \sqrt{\det[\langle e_i, e_j\rangle]_{ij}}, \qquad (1)$$

where $\langle e_i, e_j\rangle$ is the canonical inner product (the dot product) on $\mathbb{R}^n$. Here we have used the fact that the determinant of a matrix in $\mathbb{R}^{n\times n}$ is the signed volume of the parallelepiped spanned by its columns (or rows). This is a consequence of the following more general result, which is independent of the choice of basis or the normalization of μ.

Proposition 14.5. *Let $T\colon V \to V$ be a linear transformation of $V \simeq \mathbb{R}^n$. For any Haar measure μ on V and every measurable set $S \subseteq V$ we have*

$$\mu(T(S)) = |\det T|\,\mu(S). \tag{2}$$

Proof. See [8, Ex. 1.2.21]. □

If Λ is a lattice $e_1\mathbb{Z} + \cdots + e_n\mathbb{Z}$ in V, the quotient V/Λ is a compact group that we may identify with the parallelepiped $F(e_1, \ldots, e_n) \subseteq V$, which forms a set of unique coset representatives. More generally, we make the following definition.

Definition 14.6. Let Λ be a lattice in $V \simeq \mathbb{R}^n$. A *fundamental domain* for Λ is a measurable set $F \subseteq V$ such that

$$V = \bigsqcup_{\lambda\in\Lambda} (F + \lambda).$$

In other words, F is a measurable set of coset representatives for V/Λ. Fundamental domains exist: if $\Lambda = e_1\mathbb{Z} + \cdots + e_n\mathbb{Z}$ we may take the parallelepiped $F(e_1, \ldots, e_n)$.

Proposition 14.7. *Let Λ be a lattice in $V \simeq \mathbb{R}^n$ and let μ be a Haar measure on V. Every fundamental domain for Λ has the same measure, and this measure is finite and nonzero.*

Proof. Let F and G be two fundamental domains for Λ. Using the translation invariance and countable additivity of μ (note that $\Lambda \simeq \mathbb{Z}^n$ is a countable set) along with the fact that Λ is closed under negation, we obtain

$$\begin{aligned}\mu(F) = \mu(F \cap V) = \mu\left(F \cap \bigsqcup_{\lambda\in\Lambda}(G + \lambda)\right) &= \mu\left(\bigsqcup_{\lambda\in\Lambda}(F \cap (G + \lambda))\right)\\ = \sum_{\lambda\in\Lambda} \mu(F \cap (G + \lambda)) = \sum_{\lambda\in\Lambda} \mu((F - \lambda) \cap G) &= \sum_{\lambda\in\Lambda} \mu(G \cap (F + \lambda)) = \mu(G),\end{aligned}$$

where the last equality follows from the first four (swap F and G). If we fix a $\mathbb{Z}$-basis $e_1, \ldots, e_n$ for Λ, the parallelepiped $F(e_1, \ldots, e_n)$ is a fundamental domain for Λ, and its closure is compact, so $\mu(F(e_1, \ldots, e_n))$ is finite, and it is nonzero because there is an isomorphism $V \simeq \mathbb{R}^n$ that maps the closure of $F(e_1, \ldots, e_n)$ to the unit cube in $\mathbb{R}^n$ whose Lebesgue measure is nonzero (whether a set has zero measure or not does not depend on the normalization of the Haar measure and is therefore preserved by isomorphisms of locally compact groups). □

Definition 14.8. Let Λ be a lattice in $V \simeq \mathbb{R}^n$ and fix a Haar measure μ on V. The *covolume* $\operatorname{covol}(\Lambda) \in \mathbb{R}_{>0}$ of Λ is the measure $\mu(F)$ of any fundamental domain F for Λ.

Note that covolumes depend on the normalization of μ, but ratios of covolumes do not.

Proposition 14.9. *If $\Lambda' \subseteq \Lambda$ are lattices in $V \simeq \mathbb{R}^n$, then* $\operatorname{covol}(\Lambda') = [\Lambda : \Lambda']\operatorname{covol}(\Lambda)$.

Proof. Fix a fundamental domain F for Λ and a set of coset representatives S for Λ/Λ'. Then

$$F' := \bigsqcup_{\lambda\in S}(F+\lambda)$$

is a fundamental domain for Λ', and $\#S = [\Lambda:\Lambda'] = \mu(F')/\mu(F)$ is finite. We then have

$$\operatorname{covol}(\Lambda') = \mu(F') = (\#S)\mu(F) = [\Lambda:\Lambda']\operatorname{covol}(\Lambda). \qquad \square$$

Definition 14.10. Let S be a subset of a real vector space. The set S is *symmetric* if it is closed under negation, and *convex* if for all $x, y \in S$ we have $\{tx + (1-t)y : t \in [0,1]\} \subseteq S$.

Theorem 14.11 (Minkowski's Lattice Point Theorem). *Let Λ be a lattice in $V \simeq \mathbb{R}^n$ and μ a Haar measure on V. If $S \subseteq V$ is a symmetric convex measurable set that satisfies*

$$\mu(S) > 2^n \operatorname{covol}(\Lambda),$$

then S contains a nonzero element of Λ.

Proof. See Problem Set 6. $\square$

Note that the inequality in Theorem 14.11 bounds the ratio of the measures of two sets (S and a fundamental domain for Λ), and is thus independent of the choice of μ.

14.2 The canonical inner product

Let $K/\mathbb{Q}$ be a number field of degree n with r real places and s complex places; then $n = r + 2s$, by Corollary 13.9. We now want to consider the base change of K to $\mathbb{R}$ and $\mathbb{C}$:

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s,$$
$$K_{\mathbb{C}} := K \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}^n.$$

The isomorphism $K_{\mathbb{R}} \simeq \mathbb{R}^r \times \mathbb{C}^s$ follows from Theorem 13.5 and the isomorphism $K_{\mathbb{C}} \simeq \mathbb{C}^n$ follows from the fact that $\mathbb{C}$ is separably closed; see Example 4.31. We note that $K_{\mathbb{R}}$ is an $\mathbb{R}$-vector space of dimension n, thus $K_{\mathbb{R}} \simeq \mathbb{R}^n$, but this is an isomorphism of $\mathbb{R}$-vector spaces and is not an $\mathbb{R}$-algebra isomorphism unless $s = 0$.

We have a sequence of injective homomorphisms of topological rings

$$\mathcal{O}_K \hookrightarrow K \hookrightarrow K_{\mathbb{R}} \hookrightarrow K_{\mathbb{C}}, \tag{3}$$

which are defined as follows:

- the map $\mathcal{O}_K \hookrightarrow K$ is inclusion;
- the map $K \hookrightarrow K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$ is the canonical embedding $\alpha \mapsto \alpha \otimes 1$;
- the map $K_{\mathbb{R}} \simeq \mathbb{R}^r \times \mathbb{C}^s \hookrightarrow \mathbb{C}^r \times \mathbb{C}^{2s} \simeq K_{\mathbb{C}}$ embeds each factor of $\mathbb{R}^r$ in a corresponding factor of $\mathbb{C}^r$ via inclusion and each $\mathbb{C}$ in $\mathbb{C}^s$ is mapped to $\mathbb{C} \times \mathbb{C}$ in $\mathbb{C}^{2s}$ via $z \mapsto (z, \bar{z})$.

To better understand the last map, note that each $\mathbb{C}$ in $\mathbb{C}^s$ arises as $\mathbb{R}[\alpha] = \mathbb{R}[x]/(f) \simeq \mathbb{C}$ for some monic irreducible $f \in \mathbb{R}[x]$ of degree 2, but when we base-change to $\mathbb{C}$ the field $\mathbb{R}[\alpha]$ splits into the étale algebra $\mathbb{C}[x]/(x-\alpha) \times \mathbb{C}[x]/(x-\bar{\alpha}) \simeq \mathbb{C} \times \mathbb{C}$. The composition $K \hookrightarrow K_{\mathbb{R}} \hookrightarrow K_{\mathbb{C}}$ is given by the map

$$x \mapsto (\sigma_1(x), \ldots, \sigma_n(x)),$$

where $\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C}) = \{\sigma_1,\ldots,\sigma_n\}$. If we put $K = \mathbb{Q}(\alpha) := K[x]/(f)$ and let $\alpha_1,\ldots,\alpha_n \in \mathbb{C}$ be the roots of f in $\mathbb{C}$, each σ_i is the $\mathbb{Q}$-algebra homomorphism $K \to \mathbb{C}$ defined by $\alpha \mapsto \alpha_i$.

If we fix a $\mathbb{Z}$-basis for $\mathcal{O}_K$, its image under the maps in (3) is a $\mathbb{Q}$-basis for K, a $\mathbb{R}$-basis for $K_{\mathbb{R}}$, and a $\mathbb{C}$-basis for $K_{\mathbb{C}}$, all of which are vector spaces of dimension $n = [K:\mathbb{Q}]$. We may thus view the injections in (3) as inclusions of topological groups (but not rings!)

$$\mathbb{Z}^n \hookrightarrow \mathbb{Q}^n \hookrightarrow \mathbb{R}^n \hookrightarrow \mathbb{C}^n.$$

The ring of integers $\mathcal{O}_K$ is a lattice in the real vector space $K_{\mathbb{R}} \simeq \mathbb{R}^n$, which inherits an inner product from the canonical Hermitian inner product on $K_{\mathbb{C}} \simeq \mathbb{C}^n$ defined by

$$\langle z, z'\rangle := \sum_{i=1}^n z_i \bar{z}'_i \in \mathbb{C}.$$

For elements $x, y \in K \hookrightarrow K_{\mathbb{R}} \hookrightarrow K_{\mathbb{C}}$ the Hermitian inner product can be computed as

$$\langle x, y\rangle := \sum_{\sigma\in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})} \sigma(x)\overline{\sigma(y)} \in \mathbb{R}, \tag{4}$$

which is a real number because the non-real embeddings in $\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ come in complex conjugate pairs. The inner product defined in (4) agrees with the restriction of the Hermitian inner product on $K_{\mathbb{R}} \hookrightarrow K_{\mathbb{C}}$. The metric space topology it induces on $K_{\mathbb{R}}$ is the same as the Euclidean topology on $K_{\mathbb{R}} \simeq \mathbb{R}^n$ induced by the usual dot product on $\mathbb{R}^n$, but the corresponding norm $\|x\| := \langle x, x\rangle$ has a different normalization, as we now explain.

If we write elements $z \in K_{\mathbb{C}} \simeq \mathbb{C}^n$ as vectors (z_σ) indexed by the set $\sigma \in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ in some fixed order, we may identify $K_{\mathbb{R}}$ with its image in $K_{\mathbb{C}}$ as the set

$$K_{\mathbb{R}} = \{z \in K_{\mathbb{C}} : \bar{z}_\sigma = z_{\bar{\sigma}} \text{ for all } \sigma \in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})\}.$$

For real embeddings $\sigma = \bar{\sigma}$ we have $z_\sigma \in \mathbb{R} \subseteq \mathbb{C}$, and for pairs of conjugate complex embeddings $(\sigma,\bar{\sigma})$ we get the embedding $z \mapsto (z_\sigma, z_{\bar{\sigma}}) = (z_\sigma, \bar{z}_\sigma)$ of $\mathbb{C}$ into $\mathbb{C}\times\mathbb{C}$ used to defined the map $K_{\mathbb{R}} \hookrightarrow K_{\mathbb{C}}$ above. Each $z \in K_{\mathbb{R}}$ can be uniquely written in the form

$$(w_1,\ldots,w_r,\ x_1+iy_1,\ x_1-iy_1,\ \ldots,\ x_s+iy_s,\ x_s-iy_s), \tag{5}$$

with $w_i, x_j, y_j \in \mathbb{R}$. Each w_i corresponds to a z_σ with $\sigma = \bar{\sigma}$, and each (x_j+iy_j, x_j-iy_j) corresponds to a complex conjugate pair $(z_\sigma, z_{\bar{\sigma}})$ with $\sigma \ne \bar{\sigma}$. The canonical inner product on $K_{\mathbb{R}}$ can then be written as

$$\langle z, z'\rangle = \sum_{i=1}^r w_i w'_i + 2\sum_{j=1}^s (x_j x'_j + y_j y'_j).$$

Thus if we take $w_1,\ldots,w_r,x_1,y_1,\ldots,x_x,y_s$ as coordinates for $K_{\mathbb{R}} \simeq \mathbb{R}^n$ (as $\mathbb{R}$-vector spaces), in order to normalize the Haar measure μ on $K_{\mathbb{R}}$ so that it is consistent with the Lebesgue measure $\mu_{\mathbb{R}^n}$ on $\mathbb{R}^n$ we define

$$\mu(S) := 2^s \mu_{\mathbb{R}^n}(S) \tag{6}$$

for any measurable set $S \subseteq K_{\mathbb{R}}$ that we may view as a subset of $\mathbb{R}^n$ by expressing it in w_i, x_j, y_j coordinates as above. With this normalization, the identity (1) still holds when we replace $\mu_{\mathbb{R}^n}$ with μ and the dot product on $\mathbb{R}^n$ with the Hermitian inner product on $K_\mathbb{R}$, that is, for any $\mathbb{R}$-basis $e_1, \ldots, e_n$ of $K_\mathbb{R}$ we still have

$$\mu(F(e_1, \ldots, e_n)) = \sqrt{|\det[\langle e_i, e_j\rangle]_{ij}|} \tag{7}$$

Using the Hermitian inner product on $K_\mathbb{R} \subseteq K_\mathbb{C}$ rather than the dot product on $K_\mathbb{R} \simeq \mathbb{R}^n$ multiplies $2s$ of the columns in the matrix $[\langle e_i, e_j]_{ij}$ by 2, and thus multiplies the RHS by $\sqrt{2^{2s}} = 2^s$; our normalization of $\mu = 2^s\mu_{\mathbb{R}^n}$ multiplies the LHS by 2^s so that (7) still holds.

14.3 Covolumes of fractional ideals

Having fixed a normalized Haar measure μ for $K_\mathbb{R}$, we can now compute covolumes of lattices in $K_\mathbb{R} \simeq \mathbb{R}^n$. This includes not only (the image of) the ring of integers $\mathcal{O}_K$, but also any nonzero fractional ideal I of $\mathcal{O}_K$: every such I contains a nonzero principal fraction ideal $a\mathcal{O}_K$, and if $e_1, \ldots, e_n$ is a $\mathbb{Z}$-basis for $\mathcal{O}_K$ then $ae_1, \ldots, ae_n$ is a $\mathbb{Z}$-basis for $a\mathcal{O}_K$ that is an $\mathbb{R}$-basis for $K_\mathbb{R}$ that lies in I.

Recall from Remark 12.14 that the discriminant of a number field K is the integer

$$D_K := \operatorname{disc} \mathcal{O}_K := \operatorname{disc}(e_1, \ldots, e_n) \in \mathbb{Z}.$$

Proposition 14.12. *Let K be a number field. Using the normalized Haar measure on $K_\mathbb{R}$ defined in (6),*

$$\operatorname{covol}(\mathcal{O}_K) = \sqrt{|D_K|}.$$

Proof. Let $e_1, \ldots, e_n \in \mathcal{O}_K$ be a $\mathbb{Z}$-basis for $\mathcal{O}_K$, let $\operatorname{Hom}_\mathbb{Q}(K, \mathbb{C}) = \{\sigma_1, \ldots, \sigma_n\}$, and define $A := [\sigma_i(e_j)]_{ij} \in \mathbb{C}^{n\times n}$. Then $D_K = \operatorname{disc}(e_1, \ldots, e_n) = (\det A)^2$, by Proposition 12.6

Viewing $\mathcal{O}_K \hookrightarrow K_\mathbb{R}$ as a lattice in $K_\mathbb{R}$ with basis $e_1, \ldots, e_n$, we may use (7) to compute $\operatorname{covol}(\mathcal{O}_K) = \mu(F(e_1, \ldots, e_n)) = \sqrt{|\det[\langle e_i, e_j\rangle]_{ij}|}$. Applying (4) yields

$$\det[\langle e_i, e_j\rangle]_{ij} = \det\left[\textstyle\sum_k \sigma_k(e_i)\overline{\sigma_k(e_j)}\right]_{ij} = \det(A^{\mathrm{t}}\overline{A}) = (\det A)(\det \overline{A}).$$

Noting that $\det A$ is the square root of an integer (hence either real or purely imaginary), we have $\operatorname{covol}(\mathcal{O}_K)^2 = |(\det A)^2| = |D_K|$, and the proposition follows. ☐

Recall from Remark 6.13 that for number fields K we view the absolute norm

$$\begin{aligned} \mathrm{N}\colon \mathcal{I}_{\mathcal{O}_K} &\to \mathcal{I}_\mathbb{Z} \\ I &\mapsto [\mathcal{O}_K : I]_\mathbb{Z} \end{aligned}$$

as having image in $\mathbb{Q}_{>0}$ by identifying $\mathrm{N}(I) \in \mathcal{I}_\mathbb{Z}$ with a positive generator for $\mathrm{N}(I)$ (note that $\mathbb{Z}$ is a PID). Recall that $[\mathcal{O}_K : I]_\mathbb{Z}$ is a module index of $\mathbb{Z}$-lattices in the $\mathbb{Q}$-vector space K, see Definitions 6.1 and 6.5), and for ideals $I \subseteq \mathcal{O}_K$ this is just the positive integer $[\mathcal{O}_K : I]_\mathbb{Z} = [\mathcal{O}_K : I]$. When $I = (a)$ is a principal fractional ideal with $a \in K$, we may simply write $\mathrm{N}(a) := \mathrm{N}((a)) = |\mathrm{N}_{K/\mathbb{Q}}(a)|$

Corollary 14.13. *Let K be a number field and let I be a nonzero fractional ideal of $\mathcal{O}_K$. Then*

$$\operatorname{covol}(I) = \mathrm{N}(I)\sqrt{|D_K|}$$

Proof. Let $n = [K:\mathbb{Q}]$. Since $\operatorname{covol}(bI) = b^n \operatorname{covol}(I)$ and $\mathrm{N}(bI) = b^n \mathrm{N}(I)$ for any $b \in \mathbb{Z}_{>0}$, without loss of generality we may assume $I \subseteq \mathcal{O}_K$ (replace I with a suitable bI if not). Applying Propositions 14.9 and 14.12, we have

$$\operatorname{covol}(I) = [\mathcal{O}_K : I] \operatorname{covol}(\mathcal{O}_K) = \mathrm{N}(I) \operatorname{covol}(\mathcal{O}_K) = \mathrm{N}(I)\sqrt{|D_K|}$$

as claimed. □

14.4 The Minkowski bound

Theorem 14.14. Minkowski bound *Let K be a number field of degree n with s complex places. Define the Minkowski constant m_K for K as the positive real number*

$$m_K := \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|D_K|}.$$

For every nonzero fractional ideal I of $\mathcal{O}_K$ there is a nonzero $a \in I$ for which

$$\mathrm{N}(a) \le m_K \mathrm{N}(I).$$

To prove this theorem we need the following lemma.

Lemma 14.15. *Let K be a number field of degree n with r real and s complex places. For each $t \in \mathbb{R}_{>0}$, the measure of the convex symmetric set*

$$S_t := \left\{ (z_\sigma) \in K_\mathbb{R} : \sum |z_\sigma| \le t \right\} \subseteq K_\mathbb{R}$$

with respect to the normalized Haar measure μ on $K_\mathbb{R}$ is

$$\mu(S_t) = 2^r \pi^s \frac{t^n}{n!}.$$

Proof. As in (5), we may uniquely write each $z = (z_\sigma) \in K_\mathbb{R}$ in the form

$$(w_1, \ldots, w_r,\ x_1 + iy_1,\ x_1 - iy_1\ \ \ldots,\ x_s + iy_s,\ x_s - iy_s)$$

with $w_i, x_j, y_j \in \mathbb{R}$. We will have $\sum_\sigma |z_\sigma| \le t$ if and only if

$$\sum_{i=1}^r |w_i| + \sum_{j=1}^s 2\sqrt{|x_j|^2 + |y_j|^2} \le t. \tag{8}$$

We now compute the volume of this region in $\mathbb{R}^n$ by relating it to the volume of the simplex

$$U_t := \left\{ (u_1, \ldots, u_n) \in \mathbb{R}^n_{\ge 0} : u_1 + \cdots + u_n \le t \right\} \subseteq \mathbb{R}^n,$$

which is $\mu_{\mathbb{R}^n}(U_t) = t^n/n!$ (volume of the standard simplex in $\mathbb{R}^n$ scaled by a factor of t).

If we view all the w_i, x_j, y_j as fixed except the last pair (x_s, y_s), then (x_s, y_s) ranges over a disk of some radius $d \in [0, t/2]$ determined by (8). If we replace (x_s, y_s) with (u_{n-1}, u_n) ranging over the triangular region bounded by $u_{n-1} + u_n \le 2d$ and $u_{n-1}, u_n \ge 0$, we need to incorporate a factor of $\pi/2$ to account for the difference between $(2d)^2/2 = 2d^2$ and πd^2; repeat this s times. Similarly, if we hold everything but w_r fixed and replace w_r ranging over $[-d, d]$ for some $d \in [0, t]$ with u_r ranging over $[0, d]$, we need to incorporate a factor of 2 to account for this change of variable; repeat r times. We then have

$$\mu(S_t) = 2^s \mu_{\mathbb{R}^n}(S_t) = 2^s \left(\frac{\pi}{2}\right)^s 2^r \mu_{\mathbb{R}^n}(U) = 2^r \pi^s \frac{t^n}{n!}.$$ □

Proof of Theorem 14.14. Let I be a nonzero fractional ideal of $\mathcal{O}_K$. By Theorem 14.11, if we choose t so that $\mu(S_t) > 2^n \operatorname{covol}(I)$, then S_t will contain a nonzero $a \in I$. By Lemma 14.15 and Corollary 14.13, it suffices to choose t so that

$$\left(\frac{t}{n}\right)^n = \frac{n!\mu(S_t)}{n^n 2^r \pi^s} > \frac{n! 2^n}{n^n 2^r \pi^s} \operatorname{covol}(I) = \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|D_K|}\mathrm{N}(I) = m_K \mathrm{N}(I).$$

Let us now pick t so that $\left(\frac{t}{n}\right)^n > m_K\mathrm{N}(I)$. Then S_t contains $a \in I$ with $\sum_\sigma |\sigma(a)| \le t$ Recalling that the geometric mean is bounded above by the arithmetic mean, we then have

$$\mathrm{N}(a) = \left(\mathrm{N}(a)^{1/n}\right)^n = \left(\prod_\sigma |\sigma(a)|^{1/n}\right)^n \le \left(\frac{1}{n}\sum_\sigma |\sigma(a)|\right)^n \le \left(\frac{t}{n}\right)^n,$$

Taking the limit as $\left(\frac{t}{n}\right)^n \to m_K\mathrm{N}(I)$ from above yields $\mathrm{N}(a) \le m_K\mathrm{N}(I)$. $\square$

14.5 Finiteness of the ideal class group

Recall that the ideal class group $\operatorname{cl}\mathcal{O}_K$ is the quotient of the ideal group $\mathcal{I}_K$ of $\mathcal{O}_K$ by its subgroup of principal fractional ideals. We now use the Minkowski bound to prove that every ideal class $[I] \in \operatorname{cl}\mathcal{O}_K$ can be represented by an ideal $I \subseteq \mathcal{O}_K$ of small norm. It will then follow that the ideal class group is finite.

Theorem 14.16. *Let K be a number field. Every ideal class in $\operatorname{cl}\mathcal{O}_K$ contains an ideal $I \subseteq \mathcal{O}_K$ of absolute norm $\mathrm{N}(I) \le m_K$, where m_K is the Minkowski constant for K.*

Proof. Let $[J]$ be an ideal class of $\mathcal{O}_K$ represented by the nonzero fractional ideal J. By Theorem 14.14, the fractional ideal J^{-1} contains a nonzero element a for which

$$\mathrm{N}(a) \le m_K \mathrm{N}(J^{-1}) = m_K \mathrm{N}(J)^{-1},$$

and therefore $\mathrm{N}(aJ) = \mathrm{N}(a)\mathrm{N}(J) \le m_K$. We have $a \in J^{-1}$, thus $aJ \subseteq J^{-1}J = \mathcal{O}_K$, so $I = aJ$ is an $\mathcal{O}_K$-ideal in the ideal class $[J]$ with $\mathrm{N}(I) \le m_K$ as desired. $\square$

Lemma 14.17. *Let K be a number field and let M be a real number. The set of ideals $I \subseteq \mathcal{O}_K$ with $\mathrm{N}(I) \le M$ is finite.*

Proof 1. As a lattice in $K_\mathbb{R} \simeq \mathbb{R}^n$, the additive group $\mathcal{O}_K \simeq \mathbb{Z}^n$ has only finitely many subgroups I of index m for each positive integer $m \le M$, since $[\mathbb{Z}^n : I] = m$ implies

$$(m\mathbb{Z})^n \subseteq I \subseteq \mathbb{Z}^n,$$

and $(m\mathbb{Z})^n$ has finite index $m^n = [\mathbb{Z}^n : m\mathbb{Z}^n] = [\mathbb{Z} : m\mathbb{Z}]^n$ in $\mathbb{Z}^n$. $\square$

The proof of Lemma 14.17 is effective: the number of ideals $I \subseteq \mathcal{O}_K$ with $\mathrm{N}(I) \le M$ clearly cannot exceed M^{n+1}. But in fact we can give a much better bound than this.

Proof 2. Let I be an ideal of absolute norm $\mathrm{N}(I) \le M$ and let $I = \mathfrak{p}_1 \cdots \mathfrak{p}_k$ be its factorization into (not necessarily distinct) prime ideals. Then $M \ge \mathrm{N}(I) = \mathrm{N}(\mathfrak{p}_1)\cdots\mathrm{N}(\mathfrak{p}_k) \ge 2^k$, since the norm of each $\mathfrak{p}_i$ is a prime power, and in particular, at least 2. It follows that $k \le \log_2 M$ is bounded, independent of I. Each prime ideal $\mathfrak{p}$ lies above some prime $p \le M$, of which there are fewer than M, and for each prime p the number of primes $\mathfrak{p}|p$ is at most n. Thus there are fewer than $(nM)^{\log_2 M}$ ideals of norm at most M in $\mathcal{O}_K$. $\square$

Corollary 14.18. *Let K be a number field. The ideal class group of $\mathcal{O}_K$ is finite.*

Proof. By Theorem 14.16, each ideal class is represented by an ideal of norm at most m_K, and by Lemma 14.17, the number of such ideals is finite. □

Remark 14.19. For imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ it is known that the *class number* $h_K := \#\operatorname{cl}\mathcal{O}_K$ tends to infinity as $d \to \infty$ ranges over square-free integers. This was conjectured by Gauss in his *Disquisitiones Arithmeticae* [3] and proved by Heilbronn [5] in 1934; the first fully explicit lower bound was obtained by Oesterlé in 1988 [6]. This implies that there are only a finite number of imaginary quadratic fields with any particular class number. It was conjectured by Gauss that there are exactly 9 imaginary quadratic fields with class number one, but this was not proved until the 20th century by Stark [7] and Heegner [4].[1] Complete lists of imaginary quadratic fields for each class number $h_K \le 100$ are now available [9]. By contrast, Gauss predicted that infinitely many real quadratic fields should have class number 1, however this question remains completely open.[2]

Corollary 14.20. *Let K be a number field of degree n with s complex places. Then*

$$|D_K| \ge \left(\frac{n^n}{n!}\right)^2 \left(\frac{\pi}{4}\right)^{2s} > \frac{1}{e^2 n}\left(\frac{\pi e^2}{4}\right)^n.$$

Proof. If I is an ideal and $a \in I$ is nonzero, then $\mathrm{N}(a) \ge \mathrm{N}(I)$, so Theorem 14.16 implies

$$m_K = \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|D_K|} \ge 1,$$

the first inequality follows. The second uses an explicit form of Stirling's approximation,

$$n! \le e\sqrt{n}\left(\frac{n}{e}\right)^n,$$

and the fact that $2s \le n$. □

We note that $\pi e^2/4 \approx 5.8 > 1$, so the minimum value of $|D_K|$ increases exponentially with $n = [K:\mathbb{Q}]$. The lower bounds for $n \in [2,7]$ given by the corollary are listed below, along with the least value of $|D_K|$ that actually occurs. As can be seen in the table, $|D_K|$ appears to grow much faster than the corollary suggests. Better lower bounds can be proved using more advanced techniques, but a significant gap still remains.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
lower bound from Corollary 14.20	3	13	44	259	986	6267
minimum value of $\lvert D_K\rvert$	3	23	275	4511	92799	2306599

Corollary 14.21. *If K is a number field other than $\mathbb{Q}$ then $|D_K| > 1$; equivalently, there are no nontrivial unramified extensions of $\mathbb{Q}$.*

Theorem 14.22. *For every real M the set of number fields K with $|D_K| < M$ is finite.*

[1] Heegner's 1952 result [4] was essentially correct but contained some gaps that prevented it from being generally accepted until 1967 when Stark gave a complete proof in [7].

[2] In fact it is conjectured that $h_K = 1$ for approximately 75.446% of real quadratic fields with prime discriminant; this follows from the Cohen-Lenstra heuristics [2].

Proof. It follows from Corollary 14.20 that it suffices to prove this for fixed $n := [K : \mathbb{Q}]$, since for all sufficiently large n we will have $|D_K| > M$ for all number fields K of degree n.

Case 1: Let K be a totally real field (so every place $v|\infty$ is real) with $|D_K| < M$. Then $r = n$ and $s = 0$, so $K_\mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s = \mathbb{R}^n$. Consider the convex symmetric set

$$S := \{(x_1, \ldots, x_n) \in K_\mathbb{R} \simeq \mathbb{R}^n : |x_1| \le \sqrt{M} \text{ and } |x_i| < 1 \text{ for } i > 1\}$$

with measure

$$\mu(S) = 2\sqrt{M}2^{n-1} = 2^n\sqrt{M} > 2^n\sqrt{|D_K|} = 2^n \operatorname{covol}(\mathcal{O}_K).$$

By Theorem 14.11, the set S contains a nonzero $a \in \mathcal{O}_K \subseteq K \hookrightarrow K_\mathbb{R}$ that we may write as $a = (a_1, \ldots, a_n) = (\sigma_1(a), \ldots, \sigma_n(a))$, where the σ_i are the n embeddings of K into $\mathbb{C}$, all of which are real embeddings. We have

$$\mathrm{N}(a) = \left|\prod_i \sigma_i(a)\right| \ge 1,$$

since $\mathrm{N}(a)$ must be a positive integer, and $|a_2|, \ldots, |a_n| < 1$, so $|a_1| > 1 > |a_i|$ for all $i \ne 1$.

We claim that $K = \mathbb{Q}(a)$. If not, each $a_i = \sigma_i(a)$ would be repeated $[K : \mathbb{Q}(a)] > 1$ times in the vector $(a_1, \ldots, a_n)$, since there must be $[K : \mathbb{Q}(a)]$ elements of $\operatorname{Hom}_\mathbb{Q}(K, \mathbb{C})$ that fix $\mathbb{Q}(a)$, namely, those lying in the kernel of the map $\operatorname{Hom}_\mathbb{Q}(K, \mathbb{C}) \to \operatorname{Hom}_\mathbb{Q}(\mathbb{Q}(a), \mathbb{C})$ induced by restriction. But this is impossible since $a_i \ne a_1$ for $i \ne 1$.

The minimal polynomial $f \in \mathbb{Z}[x]$ of a is a monic irreducible polynomial of degree n. The roots of $f(x)$ in $\mathbb{C}$ are precisely the $a_i = \sigma_i(a) \in \mathbb{R}$, all of which are bounded by $|a_i| \le \sqrt{M}$. Each coefficient f_i of $f(x)$ is an elementary symmetric functions of its roots, hence also bounded in absolute value (certainly $|f_i| \le 2^n M^{n/2}$ for all i). The f_i are integers, so there are only finitely many possibilities for $f(x)$, hence only finitely many totally real number fields K of degree n.

Case 2: K has r real and $s > 0$ complex places, and $K_\mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s$. Now let

$$S := \{(w_1, \ldots, w_r,\ z_1,\ \ldots,\ z_s) \in K_\mathbb{R} : |z_1|^2 < c\sqrt{M} \text{ and } |w_i|, |z_j| < 1\ (j > 1)\}$$

with c chosen so that $\mu(S) > 2^n \operatorname{covol}(\mathcal{O}_K)$ (the exact value of c depends on s and n). The argument now proceeds as in case 1: we get a nonzero $a \in \mathcal{O}_K \cap S$ for which $K = \mathbb{Q}(a)$, and only a finite number of possible minimal polynomials $f \in \mathbb{Z}[x]$ for a. $\square$

Lemma 14.23. *Let K be a number field of degree n. For each prime number p we have*

$$v_p(D_K) \le n\lfloor \log_p n \rfloor + n - 1.$$

In particular, $v_p(D_K) \le n\lfloor \log_2 n \rfloor + n - 1$ for all p.

Proof. We have

$$v_p(D_K) = v_p(N_{K/\mathbb{Q}}(\mathcal{D}_{K/\mathbb{Q}})) = \sum_{\mathfrak{q}|p} f_\mathfrak{q} v_\mathfrak{q}(\mathcal{D}_{K/\mathbb{Q}})$$

where $\mathcal{D}_{K/\mathbb{Q}}$ is the different ideal and $f_\mathfrak{q}$ is the residue degree of $\mathfrak{q}|p$. Using Theorem 12.26 to bound $v_\mathfrak{q}(\mathcal{D}_{K/\mathbb{Q}})$ yields

$$v_p(D_K) \le \sum_{\mathfrak{q}|p} f_\mathfrak{q}(e_\mathfrak{q} - 1 + v_\mathfrak{q}(e_\mathfrak{q})) = n - \sum_{\mathfrak{q}|p} f_\mathfrak{q} + \sum_{\mathfrak{q}|p} f_\mathfrak{q} e_\mathfrak{q} v_p(e_\mathfrak{q}) \le n - 1 + n\lfloor \log_p n \rfloor,$$

where we have used -1 as an upper bound on $-\sum_{\mathfrak{q}|p} f_\mathfrak{q}$ and $\lfloor \log_p n \rfloor$ as an upper bound on each $v_p(e_\mathfrak{q})$ (since $e_\mathfrak{q} \le n$), and the fact that $\sum_{\mathfrak{q}|p} e_\mathfrak{q} f_\mathfrak{q} = n$ (by Theorem 5.34). $\square$

Remark 14.24. The bound in Lemma 14.23 is tight; it is achieved by $K = \mathbb{Q}[x]/(x^{p^e} - p)$, for example.

Theorem 14.25 (Hermite)**.** *Let S be a finite set of places of $\mathbb{Q}$, and let n be an integer. The number of extensions $K/\mathbb{Q}$ of degree n unramified outside of S is finite.*

Proof. By Lemma 14.23, since n is fixed, the valuation $v_p(D_K)$ is bounded for each $p \in S$ and must be zero for $p \notin S$. Thus $|D_K|$ is bounded, and the theorem then follows from Proposition 14.22. $\square$

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