18.785 Number Theory **Fall 2017**
Problem Set #1 **Due: 09/18/2017**

Description

These problems are related to the material covered in Lectures 1–3. Your solutions are to be written up in latex (you can use the latex source for the problem set as a template) and submitted as a pdf-file named `SurnamePset1.pdf` (but replace `Surname` with your surname) via e-mail to `drew@math.mit.edu` before **noon** on the date due (late problem sets will not be graded, so I encourage you to submit early). Collaboration is permitted/encouraged, but you must identify your collaborators, and any references you consulted that are not listed in the syllabus; if this does not apply to you, write **Sources consulted: none** at the top of your problem set.

The first person to spot each typo/error in any of the problem sets or lecture notes will receive 1–5 points of extra credit, depending on the severity of the error (please do report any errors you spot, even trivial typos).

Instructions: First solve the warm up problems; these do not need to be formally written up or turned in. Then pick any three of Problems 1–4 to solve and write up your answers in latex. Finally, complete Problem 5, which is a short survey whose answers will help shape future problem sets and lectures.

Problem 0. Warmup (0 points)

These warmup exercises do not need to be written up or turned in, they are provided simply to help you check your understanding.

(a) Prove the nonarchimedean "triangle equality": if $|\cdot|$ is a nonarchimedean absolute value on a field k and $|x| \neq |y|$ then $|x+y| = \max(|x|, |y|)$.

(b) Prove that an absolute value $|\cdot|$ on a field k is nonarchimedean if and only if $|n| \leq 1$ for all $n \in \mathbb{Z}_{>0}$ (view n as an element of k via the canonical ring homomorphism $\mathbb{Z} \to k$). (Hint: use the binomial theorem).

(c) Show that all absolute values on fields of positive characteristic are nonarchimedean, and all absolute values on finite fields are trivial.

(d) Write down a monic polynomial $f \in \mathbb{Z}[x]$ with $\sqrt{2} + \sqrt{3}$ as a root.

Problem 1. Absolute values on $\mathbb{Q}$ (32 points)

(a) Prove Ostrowski's Theorem: every nontrivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime $p \leq \infty$.

(b) Prove the product formula for $\mathbb{Q}$: show that $\prod_{p \leq \infty} |x|_p = 1$ for all $x \in \mathbb{Q}^\times$.

(c) Determine the archimedean absolute values on $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$.

Problem 2. Absolute values on $\mathbb{F}_q(t)$ (32 points)

For each prime $\pi \in \mathbb{F}_q[t]$ and any nonzero $f \in \mathbb{F}_q[t]$, let $v_\pi(f)$ be the largest integer for which $\pi^n | f$, equivalently, the largest n for which $f \in (\pi^n)$. For each $f/g \in \mathbb{F}_q(t)^\times$ define

$$v_\pi(f/g) := v_\pi(f) - v_\pi(g),$$

and let $v_\pi(0) := \infty$; also define $\deg 0 := -\infty$ and $\deg(f/g) := \deg f - \deg g$.

(a) For each prime $\pi \in \mathbb{F}_q[t]$, define $|r|_\pi = (q^{\deg \pi})^{-v_\pi(r)}$ for all $r \in \mathbb{F}_q(t)$. Show that $|\cdot|_\pi$ is a nonarchimedean absolute value on $\mathbb{F}_q(t)$.

(b) Define $|r|_\infty := q^{\deg r}$, for all $r \in \mathbb{F}_q(t)$. Prove that $|\cdot|_\infty$ is a nonarchimedean absolute value on $\mathbb{F}_q(t)$.

(c) Determine the residue field of $\mathbb{F}_q(t)$ with respect to $|\cdot|_\pi$; the residue field is the quotient of the valuation ring $\{x \in \mathbb{F}_q(t) : |x|_\pi \le 1\}$ by its unique maximal ideal.

(d) Describe the valuation ring $R := \{x \in \mathbb{F}_q(t) : |x|_\infty \le 1\}$ and its unique maximal ideal $\mathfrak{m}$. Then determine the residue field of $\mathbb{F}_q(t)$ with respect to $|\cdot|_\infty$.

(e) Prove Ostrowski's theorem for $\mathbb{F}_q(t)$: every nontrivial absolute value on $\mathbb{F}_q(t)$ is equivalent to $|\cdot|_\infty$ or $|\cdot|_\pi$ for some prime $\pi \in \mathbb{F}_q[t]$.

More precisely, show that if $\|\cdot\|$ is a nontrivial absolute value on $\mathbb{F}_q(t)$, either $\|t\| > 1$ and $\|\cdot\| \sim |\cdot|_\infty$, or $\|t\| \le 1$ and $\|\cdot\| \sim |\cdot|_\pi$ for some prime $\pi \in \mathbb{F}_q[t]$.

In view of (e) we now regard ∞ as a "prime" of $\mathbb{F}_q(t)$ and let π range over both monic irreducible polynomials in $\mathbb{F}_q[t]$ and ∞.

(f) Prove the product formula for $\mathbb{F}_q(t)$: show that $\prod_\pi |r|_\pi = 1$ for every $r \in \mathbb{F}_q(t)^\times$.

Problem 3. Quadratic fields (32 points)

Let $K = \mathbb{Q}(\sqrt{d})$ with $d \neq 0, 1$ a squarefree integer, and let $\mathfrak{p}$ be a nonzero prime ideal of the ring of integers $\mathcal{O}_K$ that does not divide $(2d)$.

(a) Give explicit generators for $\mathcal{O}_K$ as a $\mathbb{Z}$-lattice.

(b) Determine the index of $\mathbb{Z}[\sqrt{d}]$ in $\mathcal{O}_K$ as a function of d.

(c) Show that $\mathfrak{p}$ can be written in the form (p, α), with $(p) = \mathfrak{p} \cap \mathbb{Z}$ and $\alpha \in \mathcal{O}_K$.

(d) Show that $\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_q$ where $q = [\mathcal{O}_K : \mathfrak{p}]$ is either p or p^2, with $\mathfrak{p} = (p, \alpha)$. Give an explicit criterion in terms of p and d for when the two cases occur.

Problem 4. Dedekind domains (32 points)

Let A be a Dedekind domain with fraction field K.

(a) Describe all nonzero A-submodules of K.

(b) Describe all subrings of K containing A.

Hint: first think about the case where A is a DVR.

Problem 5. Survey (4 points)

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
9/6	Absolute values, discrete valuations				
9/11	Localizations, Dedekind domains				
9/13	Properties of Dedekind domains				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.