

## 17 The functional equation

In the previous lecture we proved that the Riemann zeta function  $\zeta(s)$  has an Euler product and an analytic continuation to the right half-plane  $\operatorname{Re}(s) > 0$ . In this lecture we complete the picture by deriving a *functional equation* that relates the values of  $\zeta(s)$  to those of  $\zeta(1-s)$ . This will then also allow us to extend  $\zeta(s)$  to a meromorphic function on  $\mathbb{C}$  that is holomorphic except for a simple pole at  $s = 1$ .

### 17.1 Fourier transforms and Poisson summation

A key tool we will use to derive the functional equation is the *Poisson summation formula*, a result from harmonic analysis that we now recall.

**Definition 17.1.** A *Schwartz function* on  $\mathbb{R}$  is a complex-valued  $C^\infty$  function  $f: \mathbb{R} \rightarrow \mathbb{C}$  that decays rapidly to zero: for all  $m, n \in \mathbb{Z}_{\geq 0}$  we have

$$\sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty,$$

where  $f^{(n)}$  denotes the  $n$ th derivative of  $f$ . The *Schwartz space*  $\mathcal{S}(\mathbb{R})$  of all Schwartz functions on  $\mathbb{R}$  is a  $\mathbb{C}$ -vector space (of infinite dimension).

**Remark 17.2.** For any  $p \in \mathbb{R}_{\geq 1}$ , the Schwartz space  $\mathcal{S}(\mathbb{R})$  is contained in the space  $L^p(\mathbb{R})$  of functions on  $f: \mathbb{R} \rightarrow \mathbb{C}$  for which the Lebesgue integral  $\int_{\mathbb{R}} |f(x)|^p dx$  exists. The space  $L^p(\mathbb{R})$  is a complete normed  $\mathbb{C}$ -vector space under the  $L^p$ -norm  $\|f\|_p := (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$ , and is thus a Banach space. The Schwartz space  $\mathcal{S}(\mathbb{R})$  is not complete under the  $L^p$  norm, but it is dense in  $L^p(\mathbb{R})$  under the subspace topology. One can equip the Schwartz space with a translation-invariant metric of its own under which it is a complete metric space (and thus a Fréchet space, since it is also locally convex), but the topology of  $\mathcal{S}(\mathbb{R})$  will not concern us here. Similar comments apply to  $\mathcal{S}(\mathbb{R}^n)$ .

It follows immediately from the definition and standard properties of the derivative that the Schwartz space  $\mathcal{S}(\mathbb{R})$  is closed under differentiation, multiplication by polynomials, sums and products, and linear change of variable. It is also closed under *convolution*: for any  $f, g \in \mathcal{S}(\mathbb{R})$  the function

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)dy$$

is also an element of  $\mathcal{S}(\mathbb{R})$ . Convolution is commutative, associative, and bilinear.

**Example 17.3.** All compactly supported functions  $C^\infty$  functions are Schwartz functions, as is the Gaussian  $g(x) := e^{-\pi x^2}$ . Non-examples include any function that does not tend to zero as  $x \rightarrow \pm\infty$  (so all nonzero polynomials), as well as functions like  $(1+x^{2n})^{-1}$ .

**Definition 17.4.** The *Fourier transform* of a Schwartz function  $f \in \mathcal{S}(\mathbb{R})$  is the function

$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx,$$

which is also a Schwartz function [1, Thm. 5.1.3]. We can recover  $f(x)$  from  $\hat{f}(y)$  via the inverse transform

$$f(x) := \int_{\mathbb{R}} \hat{f}(y)e^{+2\pi ixy}dy;$$

see [1, Thm. 5.1.9] for a proof of this fact. The maps  $f \mapsto \hat{f}$  and  $\hat{f} \mapsto f$  are thus inverse linear operators on  $\mathcal{S}(\mathbb{R})$  (they are also continuous in the metric topology of  $\mathcal{S}(\mathbb{R})$  and thus homeomorphisms).

**Remark 17.5.** The invertibility of the Fourier transform on the Schwartz space  $\mathcal{S}(\mathbb{R})$  a key motivation for its definition. For functions in  $L^1(\mathbb{R})$  (the largest space of functions for which our definition of the Fourier transform makes sense), the Fourier transform of a smooth function decays rapidly to zero, and the Fourier transform of a function that decays rapidly to zero is smooth; this leads one to consider the subspace  $\mathcal{S}(\mathbb{R})$  of smooth functions that decay rapidly to zero. One can show that  $\mathcal{S}(\mathbb{R})$  is the largest subspace of  $L^1(\mathbb{R})$  closed under multiplication by polynomials on which the Fourier transform is invertible.<sup>1</sup>

The Fourier transform changes convolutions into products, and vice versa. We have

$$\widehat{f * g} = \hat{f} \hat{g} \quad \text{and} \quad \widehat{\hat{f} g} = \hat{f} * \hat{g},$$

for all  $f, g \in \mathcal{S}(\mathbb{R})$  (see Problem Set 8). One can thus view the Fourier transform as an isomorphism of (non-unital)  $\mathbb{C}$ -algebras that sends  $(\mathcal{S}(\mathbb{R}), +, \times)$  to  $(\mathcal{S}(\mathbb{R}), +, *)$ .

**Lemma 17.6.** For all  $a \in \mathbb{R}_{>0}$  and  $f \in \mathcal{S}(\mathbb{R})$ , we have  $\widehat{f(ax)}(y) = \frac{1}{a} \hat{f}\left(\frac{y}{a}\right)$ .

*Proof.* Applying the substitution  $t = ax$  yields

$$\widehat{f(ax)}(y) = \int_{\mathbb{R}} f(ax) e^{-2\pi i xy} dx = \frac{1}{a} \int_{\mathbb{R}} f(t) e^{-2\pi i ty/a} dt = \frac{1}{a} \hat{f}\left(\frac{y}{a}\right). \quad \square$$

**Lemma 17.7.** For  $f \in \mathcal{S}(\mathbb{R})$  we have  $\frac{d}{dy} \hat{f}(y) = -2\pi i x \widehat{xf(x)}(y)$  and  $\frac{d}{dx} \widehat{f(x)}(y) = 2\pi i y \hat{f}(y)$ .

*Proof.* Noting that  $xf \in \mathcal{S}(\mathbb{R})$ , the first identity follows from

$$\frac{d}{dy} \hat{f}(y) = \frac{d}{dy} \left( \int_{\mathbb{R}} f(x) e^{-2\pi i xy} dx \right) = \int_{\mathbb{R}} f(x) (-2\pi i x) e^{-2\pi i xy} dx = -2\pi i x \widehat{xf(x)}(y),$$

since we may differentiate under the integral via dominated convergence. For the second, we note that  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ , so integration by parts yields

$$\frac{d}{dx} \widehat{f(x)}(y) = \int_{\mathbb{R}} f'(x) e^{-2\pi i xy} dx = 0 - \int_{\mathbb{R}} f(x) (-2\pi i y) e^{-2\pi i xy} dx = 2\pi i y \hat{f}(y). \quad \square$$

The Fourier transform is compatible with the inner product  $\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx$ . Indeed, we can easily derive PARSEVAL'S IDENTITY:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(y) \overline{\hat{g}(y)} e^{+2\pi i xy} dx dy = \int_{\mathbb{R}} \hat{f}(y) \overline{\hat{g}(y)} dy = \langle \hat{f}, \hat{g} \rangle,$$

which when applied to  $g = f$  yields PLANCHEREL'S IDENTITY:

$$\|f\|_2 = \langle f, f \rangle = \langle \hat{f}, \hat{f} \rangle = \|\hat{f}\|_2,$$

where  $\|f\|_2 = \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} f(x) \overline{f(x)} dx$  is the  $L^2$ -norm. For number-theoretic applications there is an analogous result due to Poisson.

<sup>1</sup>I thank Keith Conrad and Terry Tao for clarifying this point.

**Theorem 17.8** (POISSON SUMMATION FORMULA). For all  $f \in \mathcal{S}(\mathbb{R})$  we have the identity

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

*Proof.* We first note that both sums are well defined; the rapid decay property of Schwartz functions guarantees absolute convergence. Let  $F(x) := \sum_{n \in \mathbb{Z}} f(x+n)$ . Then  $F$  is a periodic  $C^\infty$ -function, so it has a Fourier series expansion

$$F(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x},$$

with Fourier coefficients

$$c_n = \int_0^1 F(x) e^{-2\pi i n x} dx = \int_0^1 \sum_{m \in \mathbb{Z}} f(x+m) e^{-2\pi i n x} dx = \int_{\mathbb{R}} f(x) e^{-2\pi i n x} dx = \hat{f}(n).$$

We then note that

$$\sum_{n \in \mathbb{Z}} f(n) = F(0) = \sum_{n \in \mathbb{Z}} \hat{f}(n). \quad \square$$

Finally, we note that the Gaussian function  $e^{-\pi x^2}$  is its own Fourier transform.

**Lemma 17.9.** Let  $g(x) := e^{-\pi x^2}$ . Then  $\hat{g}(y) = g(y)$ .

*Proof.* The function  $g(x)$  satisfies the first order ordinary differential equation

$$g' + 2\pi x g = 0, \quad (1)$$

with initial value  $g(0) = 1$ . Multiplying both sides by  $i$  and taking Fourier transforms yields

$$i(\widehat{g}' + 2\pi x \widehat{g}) = i(2\pi i x \widehat{g} - i \widehat{g}') = \widehat{g}' + 2\pi x \widehat{g} = 0,$$

via Lemma 17.7. So  $\widehat{g}$  also satisfies (1), and  $\widehat{g}(0) = \int_{\mathbb{R}} e^{-\pi x^2} dx = 1$ , so  $\widehat{g} = g$ .  $\square$

### 17.1.1 Jacobi's theta function

We now define the *theta function*<sup>2</sup>

$$\Theta(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}.$$

The sum is absolutely convergent for  $\text{Im } \tau > 0$  and thus defines a holomorphic function on the upper half plane. It is easy to see that  $\Theta(\tau)$  is periodic modulo 2, that is,

$$\Theta(\tau + 2) = \Theta(\tau),$$

but it also satisfies another functional equation.

**Lemma 17.10.** For all  $a \in \mathbb{R}_{>0}$  we have  $\Theta(ia) = \Theta(i/a)/\sqrt{a}$ .

*Proof.* Put  $g(x) := e^{-\pi x^2}$  and  $h(x) := g(\sqrt{a}x) = e^{-\pi x^2 a}$ . Lemmas 17.6 and 17.9 imply

$$\widehat{h}(y) = \widehat{g(\sqrt{a}x)}(y) = \widehat{g}(y/\sqrt{a})/\sqrt{a} = g(y/\sqrt{a})/\sqrt{a}.$$

Plugging  $\tau = ia$  into  $\Theta(\tau)$  and applying Poisson summation (Theorem 17.8) yields

$$\Theta(ia) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 a} = \sum_{n \in \mathbb{Z}} h(n) = \sum_{n \in \mathbb{Z}} \widehat{h}(n) = \sum_{n \in \mathbb{Z}} g(n/\sqrt{a})/\sqrt{a} = \Theta(i/a)/\sqrt{a} \quad \square$$

<sup>2</sup>The function  $\Theta(\tau)$  we define here is a special case of one of four parameterized families of theta functions  $\Theta_i(z : \tau)$  originally defined by Jacobi for  $i = 0, 1, 2, 3$ , which play an important role in the theory of elliptic functions and modular forms; in terms of Jacobi's notation,  $\Theta(\tau) = \Theta_3(0; \tau)$ .

### 17.1.2 Euler's gamma function

You are probably familiar with the gamma function  $\Gamma(s)$ , which plays a key role in the functional equation of not only the Riemann zeta function but many of the more general zeta functions and  $L$ -series we wish to consider. Here we recall some of its analytic properties. We begin with the definition of  $\Gamma(s)$  as a Mellin transform.

**Definition 17.11.** The *Mellin transform* of a function  $f: \mathbb{R}_{>0} \rightarrow \mathbb{C}$  is the complex function defined by

$$\mathcal{M}(f)(s) := \int_0^\infty f(t)t^{s-1}dt,$$

whenever this integral converges. It is holomorphic on  $\operatorname{Re} s \in (a, b)$  for any interval  $(a, b)$  in which the integral  $\int_0^\infty |f(t)|t^{\sigma-1}dt$  converges for all  $\sigma \in (a, b)$ .

**Definition 17.12.** The *Gamma function*

$$\Gamma(s) := \mathcal{M}(e^{-t})(s) = \int_0^\infty e^{-t}t^{s-1}dt,$$

is the Mellin transform of  $e^{-t}$ . Since  $\int_0^\infty |e^{-t}|t^{\sigma-1}dt$  converges for all  $\sigma > 0$ , the integral defines a holomorphic function on  $\operatorname{Re}(s) > 0$ .

Integration by parts yields

$$\Gamma(s) = \frac{t^s e^{-t}}{s} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-t}t^s dt = \frac{\Gamma(s+1)}{s},$$

thus  $\Gamma(s)$  has a simple pole at  $s = 0$  with residue 1 (since  $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$ ), and

$$\Gamma(s+1) = s\Gamma(s) \tag{2}$$

for  $\operatorname{Re}(s) > 0$ . Equation (2) allows us to extend  $\Gamma(s)$  to a meromorphic function on  $\mathbb{C}$  with simple poles at  $s = 0, -1, -2, \dots$ , and no other poles.

An immediate consequence of (2) is that for integers  $n > 0$  we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots 2 \cdot 1 \cdot \Gamma(1) = n!,$$

thus the gamma function can be viewed as an extension of the factorial function. The gamma function satisfies many useful identities in addition to (2), including the following.

**Theorem 17.13** (EULER'S REFLECTION FORMULA). *We have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

*as meromorphic functions on  $\mathbb{C}$  with simple poles at each integer  $s \in \mathbb{Z}$ .*

*Proof.* Let  $f(s) := \Gamma(s)\Gamma(1-s)\sin(\pi s)$ . The function  $\Gamma(s)\Gamma(1-s)$  has a simple pole at each  $s \in \mathbb{Z}$  and no other poles, while  $\sin(\pi s)$  has a zero at each  $s \in \mathbb{Z}$  and no poles, so  $f(s)$  is holomorphic on  $\mathbb{C}$ . We now note that

$$f(s+1) = \Gamma(s+1)\Gamma(-s)\sin(\pi s + \pi) = -s\Gamma(s)\Gamma(-s)\sin(\pi s) = \Gamma(s)\Gamma(1-s)\sin(\pi s) = f(s),$$

so  $f$  is periodic (with period 1). Using the substitution  $u = e^t$  we obtain

$$|\Gamma(s)| \leq \int_0^\infty |e^{-t} t^{s-1}| dt = \int_{-\infty}^\infty |e^{-e^u} e^{u(s-1)}| e^u du = \int_{-\infty}^\infty e^{u \operatorname{Re}(s) - e^u} du.$$

This implies  $|\Gamma(s)|$  is bounded on  $\operatorname{Re}(s) \in [1, 2]$ , hence on  $\operatorname{Re}(s) \in [0, 1] \cap \operatorname{Im}(s) \geq 1$ , via (2). It follows that in the strip  $\operatorname{Re}(s) \in [0, 1]$  we have

$$|f(s)| = |\Gamma(s)| |\Gamma(1-s)| |\sin(\pi s)| = O(e^{\operatorname{Im}(s)}),$$

as  $\operatorname{Im}(s) \rightarrow \infty$ , since  $|\sin(\pi s)| = \frac{1}{2} |e^{is} - e^{-is}| = O(e^{\operatorname{Im}(s)})$ . By Lemma 17.14 below,  $f(s)$  is constant. To determine the constant, as  $s \rightarrow 0$  we have  $\Gamma(s) \sim \frac{1}{s}$  and  $\sin(\pi s) \sim \pi s$ , thus

$$f(0) = \lim_{s \rightarrow 0} f(s) = \lim_{s \rightarrow 0} \Gamma(s) \Gamma(1-s) \sin(\pi s) = \lim_{s \rightarrow 0} \frac{1}{s} \cdot 1 \cdot \pi s = \pi,$$

and the theorem follows.  $\square$

**Lemma 17.14.** *Let  $f(s)$  be a holomorphic function on  $\mathbb{C}$  such that  $f(s+1) = f(s)$  and  $|f(s)| = O(e^{\operatorname{Im}(s)})$  as  $\operatorname{Im}(s) \rightarrow \infty$  in the vertical strip  $\operatorname{Re}(s) \in [0, 1]$ . Then  $f$  is constant.*

*Proof.* The function

$$g(s) = \frac{f(s) - f(a)}{\sin(\pi(s-a))}$$

is holomorphic on  $\mathbb{C}$ , since  $f(s) - f(a)$  is holomorphic and vanishes at the zeros  $a + \mathbb{Z}$  of  $\sin(\pi(s-a))$  (all of which are simple). We also have  $g(s+1) = g(s)$ , and  $|g(s)|$  is bounded on  $\operatorname{Re}(s) \in [0, 1]$ , since as  $\operatorname{Im}(s) \rightarrow \infty$  we have  $|f(s) - f(a)| = O(e^{\operatorname{Im}(s)})$  and  $|\sin(\pi(s-a))| \sim e^{\pi \operatorname{Im}(s)}$ . It follows that  $g(s)$  is bounded on  $\mathbb{C}$ , hence constant, by Liouville's theorem. We must have  $g = 0$ , since  $|g(s)| = O(e^{(1-\pi)\operatorname{Im}(s)}) = o(1)$  as  $\operatorname{Im}(s) \rightarrow \infty$ , and this implies  $f(s) = f(a)$  for all  $s \in \mathbb{C}$ .  $\square$

**Example 17.15.** Putting  $s = \frac{1}{2}$  in the reflection formula yields  $\Gamma(\frac{1}{2})^2 = \pi$ , so  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ .

**Corollary 17.16.** *The function  $\Gamma(s)$  has no zeros on  $\mathbb{C}$ .*

*Proof.* Suppose  $\Gamma(s_0) = 0$ . The RHS of the reflection formula  $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$  is never zero, since  $\sin(\pi s)$  has no poles, so  $\Gamma(1-s)$  must have a pole at  $s_0$ . Therefore  $1-s_0 \in \mathbb{Z}_{\leq 0}$ , equivalently,  $s_0 \in \mathbb{Z}_{\geq 1}$ , but then  $\Gamma(s_0) = (s_0-1)! \neq 0$ , a contradiction.  $\square$

### 17.1.3 Completing the zeta function

Let us now consider the function

$$F(s) := \pi^{-s} \Gamma(s) \zeta(2s),$$

which is a meromorphic on  $\mathbb{C}$  and holomorphic on  $\operatorname{Re}(s) > 1/2$ . In the region  $\operatorname{Re}(s) > 1/2$  we have an absolutely convergent sum

$$F(s) = \pi^{-s} \Gamma(s) \sum_{n \geq 1} n^{-2s} = \sum_{n \geq 1} (\pi n^2)^{-s} \Gamma(s) = \sum_{n \geq 1} \int_0^\infty (\pi n^2)^{-s} t^{s-1} e^{-t} dt,$$

and the substitution  $t = \pi n^2 y$  with  $dt = \pi n^2 dy$  yields

$$F(s) = \sum_{n \geq 1} \int_0^\infty (\pi n^2)^{-s} (\pi n^2 y)^{s-1} e^{-\pi n^2 y} \pi n^2 dy = \sum_{n \geq 1} \int_0^\infty y^{s-1} e^{-\pi n^2 y} dy.$$

By the Fubini-Tonelli theorem, we can swap the sum and the integral to obtain

$$F(s) = \int_0^\infty y^{s-1} \sum_{n \geq 1} e^{-\pi n^2 y} dy.$$

We have  $\Theta(iy) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} = 1 + 2 \sum_{n \geq 1} e^{-\pi n^2 y}$ , thus

$$\begin{aligned} F(s) &= \frac{1}{2} \int_0^\infty y^{s-1} (\Theta(iy) - 1) dy \\ &= \frac{1}{2} \left( \int_0^1 y^{s-1} \Theta(iy) dy - \frac{1}{s} + \int_1^\infty y^{s-1} (\Theta(iy) - 1) dy \right) \end{aligned}$$

We now focus on the first integral on the RHS. The change of variable  $t = \frac{1}{y}$  yields

$$\int_0^1 y^{s-1} \Theta(iy) dy = \int_\infty^1 t^{1-s} \Theta(i/t) (-t^{-2}) dt = \int_1^\infty t^{-s-1} \Theta(i/t) dt.$$

By Lemma 17.10,  $\Theta(i/t) = \sqrt{t} \Theta(it)$ , and adding  $-\int_1^\infty t^{-s-1/2} dt + \int_1^\infty t^{-s-1/2} dt = 0$  yields

$$\begin{aligned} &= \int_1^\infty t^{-s-1/2} (\Theta(it) - 1) dt + \int_1^\infty t^{-s-1/2} dt \\ &= \int_1^\infty t^{-s-1/2} (\Theta(it) - 1) dt - \frac{1}{1/2 - s}. \end{aligned}$$

Plugging this back into our equation for  $F(s)$  we obtain the identity

$$F(s) = \frac{1}{2} \int_1^\infty (y^{s-1} + y^{-s-1/2}) (\Theta(iy) - 1) dy - \frac{1}{2s} - \frac{1}{1-2s},$$

valid on  $\operatorname{Re}(s) > 1/2$ . We now observe that  $F(s) = F(\frac{1}{2} - s)$  for  $s \neq 0, \frac{1}{2}$ , which allows us to analytically extend  $F(s)$  to a meromorphic function on  $\mathbb{C}$  with poles only at  $s = 0, 1/2$ . Replacing  $s$  with  $s/2$  leads us to define the *completed zeta function*

$$Z(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \tag{3}$$

which is meromorphic on  $\mathbb{C}$  and satisfies the *functional equation*

$$Z(s) = Z(1-s). \tag{4}$$

It has simple poles at 0 and 1 (and no other poles). The only zeros of  $Z(s)$  on  $\operatorname{Re}(s) > 0$  are the zeros of  $\zeta(s)$ , since by Corollary 17.16, the gamma function  $\Gamma(s)$  has no zeros (and neither does  $\pi^{-s/2}$ ). Thus the zeros of  $Z(s)$  on  $\mathbb{C}$  all lie in the critical strip  $0 < \operatorname{Re}(s) < 1$ .

The functional equation also allows us to analytically extend  $\zeta(s)$  to a meromorphic function on  $\mathbb{C}$  whose only pole is a simple pole at  $s = 1$ ; the pole of  $Z(s)$  at  $s = 0$  comes from the pole of  $\Gamma(s/2)$  at  $s = 0$ . The function  $\Gamma(s/2)$  also has poles at  $-2, -4, \dots$  where  $Z(s)$  does not, so our extended  $\zeta(s)$  must have zeros at  $-2, -4, \dots$ . These are *trivial zeros*;

all the interesting zeros of  $\zeta(s)$  lie in the critical strip and are conjectured to lie only on the critical line  $\operatorname{Re}(s) = 1/2$  (this is the Riemann hypothesis).

We can compute  $\zeta(0)$  using the functional equation. From (3) and (4) we have

$$\zeta(s) = \frac{Z(s)}{\pi^{-s/2}\Gamma(\frac{s}{2})} = \frac{Z(1-s)}{\pi^{-s/2}\Gamma(\frac{s}{2})} = \frac{\pi^{\frac{(s-1)}{2}}\Gamma(\frac{1-s}{2})}{\pi^{-s/2}\Gamma(\frac{s}{2})}\zeta(1-s) = \frac{\pi^{s-1/2}\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s), \quad (5)$$

We know that  $\zeta(s)$  has a simple pole with residue 1 at  $s = 1$ , so

$$1 = \lim_{s \rightarrow 1^+} (s-1)\zeta(s) = \lim_{s \rightarrow 1^+} \frac{(s-1)\pi^{s-1/2}\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s).$$

When  $s = 1$ , the denominator on the RHS is  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ , which cancels  $\pi^{1-1/2} = \sqrt{\pi}$  in the numerator. Using  $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$  to shift  $\Gamma(\frac{1-s}{2})$  in the numerator yields

$$1 = \lim_{s \rightarrow 1^+} (s-1)\frac{2}{1-s}\Gamma(\frac{3-s}{2})\zeta(1-s) = -2\Gamma(1)\zeta(0) = -2\zeta(0).$$

Thus  $\zeta(0) = -1/2$ .

Using the reflection formula to replace  $\Gamma(\frac{s}{2}) = \pi/(\Gamma(\frac{2-s}{2})\sin(\frac{\pi s}{2}))$  in (5), we have

$$\zeta(s) = \pi^{s-3/2}\Gamma(\frac{1-s}{2})\Gamma(\frac{2-s}{2})\sin(\frac{\pi s}{2})\zeta(1-s).$$

Applying the [duplication formula](#)  $\Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$  with  $z = \frac{1-s}{2}$  then yields

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s), \quad (6)$$

which is how one often sees the functional equation for  $\zeta(s)$  written.

## 17.2 Gamma factors and a holomorphic zeta function

If we write out the Euler product for the completed zeta function, we have

$$Z(s) = \pi^{-s/2}\Gamma(\frac{s}{2}) \cdot \prod_p (1-p^{-s})^{-1}.$$

One should think of this as a product over the places of the field  $\mathbb{Q}$ ; the leading factor

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(\frac{s}{2})$$

that distinguishes the completed zeta function  $Z(s)$  from  $\zeta(s)$  corresponds to the real archimedean place of  $\mathbb{Q}$ . When we discuss Dedekind zeta functions in a later lecture we will see that there are gamma factors  $\Gamma_{\mathbb{R}}$  and  $\Gamma_{\mathbb{C}}$  associated to each of the real and complex places of a number field.

If we insert an additional factor of  $\binom{s}{2} := \frac{s(s-1)}{2}$  in  $Z(s)$  we can remove the poles at 0 and 1, yielding a function  $\xi(s)$  holomorphic on  $\mathbb{C}$ . This yields Riemann's seminal result.

**Theorem 17.17** (ANALYTIC CONTINUATION II). *The function*

$$\xi(s) := \binom{s}{2}\Gamma_{\mathbb{R}}(s)\zeta(s)$$

*is holomorphic on  $\mathbb{C}$  and satisfies the functional equation*

$$\xi(s) = \xi(1-s).$$

*The zeros of  $\xi(s)$  all lie in the critical strip  $0 < \operatorname{Re}(s) < 1$ .*

**Remark 17.18.** We will usually work with  $Z(s)$  and deal with the poles rather than making it holomorphic by introducing additional factors; some authors use  $\xi(s)$  to denote our  $Z(s)$ .

## References

- [1] Elias M. Stein and Rami Shakarchi, *Fourier analysis: an introduction*, Princeton University Press, 2007.