10 Extensions of complete DVRs

Recall that in our AKLB setup, $A$ is a Dedekind domain with fraction field $K$, the field $L$ is a finite separable extension of $K$, and $B$ is the integral closure of $A$ in $L$; as we proved in Theorem 5.22, this implies that $B$ is also a Dedekind domain (with $L$ as its fraction field).

We now want to consider the special case where $A$ is a complete DVR; in this case $B$ is also a complete DVR, but this will take a little bit of work to prove. We first show that $B$ is a DVR.

**Theorem 10.1.** Assume AKLB and that $A$ is a complete DVR with maximal ideal $p$. Then $B$ is a DVR whose maximal ideal $q$ is necessarily the unique prime above $p$.

**Proof.** We first show that $\#\{q|p\} = 1$. At least one prime $q$ of $B$ lies above $p$, since the factorization of $pB \subseteq B$ is non-trivial. Now suppose for the sake of contradiction that $q_1, q_2 \in \{q|p\}$ with $q_1 \neq q_2$. Choose $b \in q_1 - q_2$ and consider the ring $A[b] \subseteq B$. The ideals $q_1 \cap A[b]$ and $q_2 \cap A[b]$ are distinct prime ideals of $A[b]$ containing $pA[b]$, and both are maximal, since they are nonzero and $\dim A[b] = \dim A = 1$ (note that $A[b] \subseteq B$ is integral over $A$ and therefore has the same dimension). The quotient ring $A[b]/pA[b]$ thus has at least two maximal ideals. Let $f \in A[x]$ be the minimal polynomial of $b$ over $K$, and let $\bar{f} \in k[x]$ be its reduction to the residue field $A/p$. We have

$$\frac{(A/p)[x]}{(f)} \simeq \frac{A[x]}{(p, f)} \simeq \frac{A[b]}{pA[b]},$$

thus the ring $(A/p)[x]/(\bar{f})$ has at least two maximal ideals, which implies that $\bar{f}$ is divisible by two distinct irreducible polynomials (because $(A/p)[x]$ is a PID). We can thus factor $f = gh$ with $g$ and $h$ coprime. By Hensel’s Lemma 9.19, we can lift this to a non-trivial factorization $f = gh$ of $f$ in $A[x]$, contradicting the irreducibility of $f$.

Every maximal ideal of $B$ lies above a maximal ideal of $A$, but $A$ has only the maximal ideal $p$ and $\#\{q|p\} = 1$, so $B$ has a unique (nonzero) maximal ideal $q$. Thus $B$ is a local Dedekind domain, hence a local PID, and not a field, so $B$ is a DVR, by Theorem 1.15. □

**Remark 10.2.** The assumption that $A$ is complete is necessary. For example, if $A$ is the DVR $\mathbb{Z}(5)$ with fraction field $K = \mathbb{Q}$ and we take $L = \mathbb{Q}(i)$, then the integral closure of $A$ in $L$ is $B = \mathbb{Z}(5)[i]$, which is a PID but not a DVR: the ideals $(1+2i)$ and $(1-2i)$ are both maximal (and not equal). But if we take completions we get $A = \mathbb{Z}_5$ and $K = \mathbb{Q}_5$, and now $L = \mathbb{Q}_5(i) = \mathbb{Q}_5 = K$, since $x^2 + 1$ has a root in $\mathbb{F}_5 \simeq \mathbb{Z}_5/5\mathbb{Z}_5$ that we can lift to $\mathbb{Z}_5$ via Hensel’s lemma; thus if we complete $A$ then $B = A$ is a DVR as required.

**Definition 10.3.** Let $K$ be a field with absolute value $|\ |$ and let $V$ be a $K$-vector space. A *norm* on $V$ is a function $\|\| : V \to \mathbb{R}_{\geq 0}$ such that

- $\|v\| = 0$ if and only if $v = 0$.
- $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in K$ and $v \in V$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Each norm $\|\|$ induces a topology on $V$ via the distance metric $d(v, w) := \|v - w\|$.

**Example 10.4.** Let $V$ be a $K$-vector space with basis $(e_i)$, and for $v \in V$ let $v_i \in K$ denote the coefficient of $e_i$ in $v = \sum_i v_i e_i$. The sup-norm $\|v\|_\infty := \sup\{|v_i|\}$ is a norm on $V$ (so

Andrew V. Sutherland
every vector space has at least one norm). If \( V \) is also a \( K \)-algebra, an absolute value \( \| \| \) on \( V \) (as a ring) is a norm on \( V \) (as a \( K \)-vector space) if and only if it extends the absolute value on \( K \) (fix \( v \neq 0 \) and note that \( \| \lambda \| \| v \| = \| \lambda v \| = | \lambda | \| v \| \iff \| \lambda \| = | \lambda | \)).

**Proposition 10.5.** Let \( V \) be a vector space of finite dimension over a complete field \( K \). Every norm on \( V \) induces the same topology, in which \( V \) is a complete metric space.

**Proof.** See Problem Set 5.

**Theorem 10.6.** Let \( A \) be a complete DVR with fraction field \( K \), maximal ideal \( \mathfrak{p} \), discrete valuation \( v_\mathfrak{p} \), and absolute value \( |x|_\mathfrak{p} := e^{v_\mathfrak{p}(x)} \), with \( 0 < c < 1 \). Let \( L/K \) be a finite extension of degree \( n \). The following hold.

(i) There is a unique absolute value \( |x| := |N_{L/K}(x)|_\mathfrak{p}^{1/n} \) on \( L \) that extends \( |x|_\mathfrak{p} \);

(ii) The field \( L \) is complete with respect to \( |x| \), and its valuation ring \( \{ x \in L : |x| \leq 1 \} \) is equal to the integral closure \( B \) of \( A \) in \( L \);

(iii) If \( L/K \) is separable then \( B \) is a complete DVR whose maximal ideal \( \mathfrak{q} \) induces

\[
|x| = |x|_\mathfrak{q} := c^{\frac{1}{e_\mathfrak{q}}v_\mathfrak{q}(x)},
\]

where \( e_\mathfrak{q} \) is the ramification index of \( \mathfrak{q} \), that is, \( \mathfrak{p}B = \mathfrak{q}^{e_\mathfrak{q}} \).

**Proof.** Assuming for the moment that \( | \| \) is actually an absolute value (which is not obvious!), for any \( x \in K \) we have

\[
|x| = |N_{L/K}(x)|_\mathfrak{p}^{1/n} = |x^n|_\mathfrak{p}^{1/n} = |x|_\mathfrak{p},
\]

so \( | \| \) extends \( |x|_\mathfrak{p} \) and is therefore a norm on \( L \). The fact that \( | \| \) \( |x|_\mathfrak{p} \) is nontrivial means that \( |x|_\mathfrak{p} \neq 1 \) for some \( x \in K^\times \), and \( |x|^a = |x|_\mathfrak{p} = |x| \) only for \( a = 1 \), which implies that \( | \| \) is the unique absolute value in its equivalence class extending \( |x|_\mathfrak{p} \). Every norm on \( L \) induces the same topology (by Proposition 10.5), so \( | \| \) is the only absolute value on \( L \) that extends \( |x|_\mathfrak{p} \).

We now show \( | \| \) is an absolute value. Clearly \( |x| = 0 \iff x = 0 \) and \( | \| \) is multiplicative; we only need to check the triangle inequality. It suffices to show \( |x| \leq 1 \Rightarrow |x+1| \leq |x|+1 \), since we always have \( |y+z| = |y| |y/z+1| \) and \( |y+z| = |z+(y/z+1)| \), and without loss of generality we assume \( |y| \leq |z| \). In fact the stronger implication \( |x| \leq 1 \Rightarrow |x+1| \leq 1 \) holds:

\[
|x| \leq 1 \iff |N_{L/K}(x)|_\mathfrak{p} \leq 1 \iff N_{L/K}(x) \in A \iff x \in B \iff x+1 \in B \iff |x+1| \leq 1.
\]

The first biconditional follows from the definition of \( | \| \), the second follows from the definition of \( | \|_\mathfrak{p} \), the third is Corollary 9.21, the fourth is obvious, and the fifth follows from the first three after replacing \( x \) with \( x + 1 \). This completes the proof of (i), and also proves (ii).

We now assume \( L/K \) is separable. Then \( B \) is a DVR, by Theorem 10.1, and it is complete because it is the valuation ring of \( L \). Let \( \mathfrak{q} \) be the unique maximal ideal of \( B \). The valuation \( v_\mathfrak{q} \) extends \( v_\mathfrak{p} \) with index \( e_\mathfrak{q} \), by Theorem 8.20, so \( v_\mathfrak{q}(x) = e_\mathfrak{q}v_\mathfrak{p}(x) \) for \( x \in K^\times \).

We have \( 0 < c^{1/e_\mathfrak{q}} < 1 \), so \( |x|_\mathfrak{q} := (c^{1/e_\mathfrak{q}})^{v_\mathfrak{p}(x)} \) is an absolute value on \( L \) induced by \( v_\mathfrak{q} \). To show it is equal to \( | \| \), it suffices to show that it extends \( |x|_\mathfrak{p} \), since we already know that \( | \| \) is the unique absolute value on \( L \) with this property. For \( x \in K^\times \) we have

\[
|x|_\mathfrak{q} = c^{\frac{1}{e_\mathfrak{q}}v_\mathfrak{q}(x)} = c^{\frac{1}{e_\mathfrak{q}}e_\mathfrak{q}v_\mathfrak{p}(x)} = c^{v_\mathfrak{p}(x)} = |x|_\mathfrak{p},
\]

and the theorem follows. 

\[
18.785 \text{ Fall 2017, Lecture #10, Page 2}
\]
Remark 10.7. The transitivity of $N_{L/K}$ in towers (Corollary 4.52) implies that we can uniquely extend the absolute value on the fraction field $K$ of a complete DVR to an algebraic closure $\overline{K}$. In fact, this is another form of Hensel’s lemma in the following sense: one can show that a (not necessarily discrete) valuation ring $A$ is Henselian if and only if the absolute value of its fraction field $K$ can be uniquely extended to $\overline{K}$; see [4, Theorem 6.6].

Corollary 10.8. Assume $AKLB$ and that $A$ is a complete DVR with maximal ideal $p$ and let $q/p$. Then $v_q(x) = \frac{1}{e_q}v_p(N_{L/K}(x))$ for all $x \in L$.

Proof. $v_p(N_{L/K}(x)) = v_p(N_{L/K}((x))) = v_p(N_{L/K}(q^{v_q(x)})) = v_p(p^{f_qv_q(x)}) = f_qv_q(x)$.

Remark 10.9. One can generalize the notion of a discrete valuation to a valuation, a surjective homomorphism $v: K^\times \to \Gamma$, in which $\Gamma$ is a (totally) ordered abelian group and $v(x+y) \leq \max(v(x), v(y))$: we extend $v$ to $K$ by defining $v(0) = \infty$ to be strictly greater than any element of $\Gamma$. In the $AKLB$ setup with $A$ a complete DVR, one can then define a valuation $v(x) = \frac{1}{e_q}v_q(x)$ with image $\frac{1}{e_q}\mathbb{Z}$ that restricts to the discrete valuation $v_p$ on $K$. The valuation $v$ then extends to a valuation on $\overline{K}$ with $\Gamma = Q$. Some texts take this approach, but we will generally stick with discrete valuations (so our absolute value on $L$ restricts to $K$, but our discrete valuations on $L$ do not restrict to discrete valuations on $K$, they extend them with index $e_q$).

Remark 10.10. Recall that a valuation ring is an integral domain $A$ with fraction field $K$ such that for every $x \in K^\times$ either $x \in A$ or $x^{-1} \in A$ (possibly both). As you will show on Problem Set 6, if $A$ is a valuation ring, then there exists a valuation $v: K \to \Gamma \cup \{\infty\}$ for some totally ordered abelian group $\Gamma$ such that $A = \{x \in K : v(x) \geq 0\}$ is the valuation ring of $K$ with respect to this valuation.

In our $AKLB$ setup, if $A$ is a complete DVR with maximal ideal $p$ then $B$ is a complete DVR with maximal ideal $q/p$ and the formula $[L : K] = \sum_{p/q} e_qf_q$ given by Theorem 5.32 has only one term $e_qf_q$. We now simplify matters even further by reducing to the two extreme cases $f_q = 1$ (a totally ramified extension) and $e_q = 1$ (an unramified extension, provided that the residue field extension is separable).\footnote{Recall from Definition 5.34 that separability of the residue field extension is part of the definition of an unramified extension. If the residue field is perfect (as when $K$ is a local field, for example), the residue field extension is automatically separable, but in general it need not be, even when $L/K$ is unramified.}

10.1 The Dedekind-Kummer theorem in a local setting

Recall that the Dedekind-Kummer theorem (Theorem 6.14) allows us to factor primes in our $AKLB$ setting by factoring polynomials over the residue field, provided that $B$ is monogenic (of the form $A[\alpha]$ for some $\alpha \in B$), or the prime of interest does not contain the conductor. We now show that in the special case where $A$ and $B$ are DVRs and the residue field extension is separable, $B$ is always monogenic; this holds, for example, whenever $K$ is a local field. To prove this, we first recall a form of Nakayama’s lemma.

Lemma 10.11 (Nakayama’s Lemma). Let $A$ be a local ring with maximal ideal $p$, and let $M$ be a finitely generated $A$-module. If the images of $x_1, \ldots, x_n \in M$ generate $M/pM$ as an $(A/p)$-vector space then $x_1, \ldots, x_n$ generate $M$ as an $A$-module.

Proof. See [1, Corollary 4.8b].
Before proving our theorem on local monogenicity, we record a few corollaries of Nakayama’s Lemma that will be useful later.

**Corollary 10.12.** Let $A$ be a local noetherian ring with maximal ideal $p$, let $g \in A[x]$, and let $B := A[x]/(g(x))$. Every maximal ideal $m$ of $B$ contains the ideal $pB$.

*Proof.* Suppose not. Then $m + pB = B$ for some maximal ideal $m$ of $B$. The ring $B$ is finitely generated over the noetherian ring $A$, hence a noetherian $A$-module, so its $A$-submodules are all finitely generated. Let $z_1, \ldots, z_n$ be $A$-module generators for $m$. Every coset of $pB$ in $B$ can be written as $z + pB$ for some $A$-linear combination $z$ of $z_1, \ldots, z_n$, so the images of $z_1, \ldots, z_n$ generate $B/pB$ as an $(A/p)$-vector space. By Nakayama’s lemma, $z_1, \ldots, z_n$ generate $B$, in which case $m = B$, a contradiction. \hfill $\square$

As a corollary, we immediately obtain a local version of the Dedekind-Kummer theorem that does not even require $A$ and $B$ to be Dedekind domains.

**Corollary 10.13.** Let $A$ be a local noetherian ring with maximal ideal $p$, let $g \in A[x]$ be a polynomial with reduction $\bar{g} \in (A/p)[x]$, and let $\alpha$ be the image of $x$ in the ring $B := A[x]/(g(x)) = A[\alpha]$. The maximal ideals of $B$ are $(p, g_i(\alpha))$, where $g_1, \ldots, g_m \in A[x]$ are lifts of the distinct irreducible polynomials $g_i \in (A/p)[x]$ that divide $\bar{g}$.

*Proof.* By Corollary 10.12, the quotient map $B \to B/pB$ gives a one-to-one correspondence between maximal ideals of $B$ and maximal ideals of $B/pB$, and we have

$$B/pB \cong \frac{A[x]}{(p, g(x))} \cong \frac{(A/p)[x]}{\bar{g}(x)}.$$

Each maximal ideal of $(A/p)[x]/(\bar{g}(x))$ is the reduction of an irreducible divisor of $\bar{g}$, hence one of the $g_i$ (because $(A/p)[x]$ is a PID). The corollary follows. \hfill $\square$

**Theorem 10.14.** Assume $AKLB$, with $A$ and $B$ DVRs with residue fields $k := A/p$ and $\ell := B/q$. If $l/k$ is separable then $B = A[\alpha]$ for some $\alpha \in B$; if $L/K$ is unramified this holds for any $\alpha \in B$ whose image generates the residue field extension $l/k$.

*Proof.* Let $pB = q^e$ be the factorization of $pB$ and let $f = [l : k]$ be the residue field degree, so that $ef = n := [L : K]$. The extension $l/k$ is separable, so we may apply the primitive element theorem to write $l = k(\alpha_0)$ for some $\alpha_0 \in l$ whose minimal polynomial $\bar{g}$ is separable of degree $f$ (so $\bar{g}(\alpha_0) = 0$ and $\bar{g}'(\alpha_0) \neq 0$). Let $\alpha_0$ be any lift of $\bar{\alpha}_0$ to $B$, and let $g \in A[x]$ be a monic lift of $\bar{g}$ chosen so that $v_q(g(\alpha_0)) > 1$ and $v_q(g'(\alpha_0)) = 0$. This is possible since $g(\alpha_0) \equiv \bar{g}(\alpha_0) \equiv 0 \mod q$, so $v_q(g(\alpha_0)) \geq 1$ and if equality holds we can replace $g$ by $g - g(\alpha_0)$ without changing $g'(\alpha_0) \equiv \bar{g}'(\alpha_0) \neq 0 \mod q$. Now let $\pi_0$ be any uniformizer for $B$ and let $\alpha := \alpha_0 + \pi_0 \in B$ (so $\alpha \equiv \alpha_0 \mod q$) Writing $g(x + \pi_0) = g(x) + \pi_0 g'(x) + \pi_0^2 h(x)$ for some $h \in A[x]$ via Lemma 9.11, we have

$$v_q(g(\alpha)) = v_q(g(\alpha_0 + \pi_0)) = v_q(g(\alpha_0) + \pi_0 g'(\alpha_0) + \pi_0^2 h(\alpha_0)) = 1,$$

so $\pi := g(\alpha)$ is also a uniformizer for $B$.

We now claim $B = A[\alpha]$, equivalently, that $1, \alpha, \ldots, \alpha^{n-1}$ generate $B$ as an $A$-module. By Nakayama’s lemma, it suffices to show that the reductions of $1, \alpha, \ldots, \alpha^{n-1}$ span $B/pB$ as an $k$-vector space. We have $p = q^e$, so $pB = (\pi^e)$. We can represent each element of $B/pB$ as a coset $b + pB = b_0 + b_1 \pi + b_2 \pi^2 \cdots + b_{e-1} \pi^{e-1} + pB$. 

18.785 Fall 2017, Lecture #10, Page 4
where \( b_0, \ldots, b_{e-1} \) are determined up to equivalence modulo \( \pi B \). Now \( 1, \alpha, \ldots, \alpha^{f-1} \) are a basis for \( B/\pi B = B/q \) as a \( k \)-vector space, and \( \pi = g(\alpha) \), so we can rewrite this as

\[
b + pB = (a_0 + a_1 \alpha + \cdots + a_{f-1} \alpha^{f-1}) \\
+ (a_f + a_{f+1} \alpha + \cdots + a_{2f-1} \alpha^{f-1})g(\alpha) \\
+ \cdots \\
+ (a_{ef-f+1} + a_{ef-f+2} \alpha + \cdots + a_{ef-1} \alpha^{f-1})g(\alpha)^{e-1} + pB.
\]

Since \( \deg g = f \), and \( n = ef \), this expresses \( b + pB \) in the form \( b' + pB \) with \( b' \) in the \( A \)-span of \( 1, \ldots, \alpha^{n-1} \). Thus \( B = A[\alpha] \).

We now note that if \( L/K \) is unramified then \( l/k \) is separable (this is part of the definition of unramified), and \( e = 1 \), \( f = n \), in which case there is no need to require \( g(\alpha) \) to be a uniformizer and we can just take \( \alpha = \alpha_0 \) to be any lift of any \( \bar{\alpha}_0 \) that generates \( l \) over \( k \). 

### 10.2 Unramified extensions of a complete DVR

Let \( A \) be a complete DVR with fraction field \( K \) and residue field \( k \). Associated to any finite unramified extension of \( L/K \) of degree \( n \) is a corresponding finite separable extension of residue fields \( l/k \) of the same degree \( n \). Given that the extensions \( L/K \) and \( l/k \) are finite separable extensions of the same degree, we might wonder how they are related. More precisely, if we fix \( K \) with residue field \( k \), what is the relationship between finite unramified extensions \( L/K \) of degree \( n \) and finite separable extensions \( l/k \) of degree \( n' \)? Each \( L/K \) uniquely determines a corresponding \( l/k \), but what about the converse?

This question has a surprisingly nice answer. The finite unramified extensions \( L \) of \( K \) form a category \( C_{K}^{\text{unr}} \) whose morphisms are \( K \)-algebra homomorphisms, and the finite separable extensions \( l \) of \( k \) form a category \( C_{k}^{\text{sep}} \) whose morphisms are \( k \)-algebra homomorphisms. These two categories are equivalent.

**Theorem 10.15.** Let \( A \) be a complete DVR with fraction field \( K \) and residue field \( k := A/p \). The categories \( C_{K}^{\text{unr}} \) and \( C_{k}^{\text{sep}} \) are equivalent via the functor \( F : C_{K}^{\text{unr}} \to C_{k}^{\text{sep}} \) that sends each unramified extension \( L \) of \( K \) to its residue field \( l \), and each \( K \)-algebra homomorphism \( \varphi : L_1 \to L_2 \) to the \( k \)-algebra homomorphism \( \bar{\varphi} : l_1 \to l_2 \) defined by \( \bar{\varphi}(\bar{\alpha}) := \bar{\varphi}(\bar{\alpha}) \), where \( \alpha \) is any lift of \( \bar{\alpha} \in l_1 := B_1/q_1 \) to \( B_1 \) and \( \bar{\varphi}(\alpha) \) is the reduction of \( \varphi(\alpha) \in B_2 \) to \( l_2 := B_2/q_2 \); here \( q_1, q_2 \) are the maximal ideals of the valuation rings \( B_1, B_2 \) of \( L_1, L_2 \), respectively.

In particular, \( F \) gives a bijection between the isomorphism classes in \( C_{K}^{\text{unr}} \) and \( C_{k}^{\text{sep}} \), and if \( L_1, L_2 \) and have residue fields \( l_1, l_2 \) then \( F \) induces a bijection of finite sets

\[
\text{Hom}_K(L_1, L_2) \cong \text{Hom}_k(l_1, l_2).
\]

**Proof.** Let us first verify that \( F \) is well-defined. It is clear that it maps finite unramified extensions \( L/K \) to finite separable extension \( l/k \), but we should check that the map on morphisms does not depend on the lift \( \alpha \) of \( \bar{\alpha} \) we pick. So let \( \varphi : L_1 \to L_2 \) be a \( K \)-algebra homomorphism, and for \( \bar{\alpha} \in l_1 \), let \( \alpha \) and \( \alpha' \) be two lifts of \( \bar{\alpha} \) to \( B_1 \). Then \( \alpha - \alpha' \in q_1 \), and this implies that \( \varphi(\alpha - \alpha') \in \varphi(q_1) = \varphi(B_1) \cap q_2 \subseteq q_2 \), and therefore \( \bar{\varphi}(\alpha) = \bar{\varphi}(\alpha') \). The identity \( \varphi(q_1) = \varphi(B_1) \cap q_2 \subseteq q_2 \) follows from the fact that \( \varphi \) restricts to an injective ring homomorphism \( B_1 \to B_2 \) and \( B_2/\varphi(B_1) \) is a finite extension of DVRs in which \( q_2 \) lies over the prime \( \varphi(q_1) \) of \( \varphi(B_1) \). It’s easy to see that \( F \) sends identity morphisms to identity morphisms and that it is compatible with composition, so we have a well-defined functor.

To show that \( F \) is an equivalence of categories we need to prove two things:
• $\mathcal{F}$ is essentially surjective: each separable $l/k$ is isomorphic to the residue field of some unramified $L/K$

• $\mathcal{F}$ is full and faithful: the induced map $\text{Hom}_K(L_1, L_2) \to \text{Hom}_k(l_1, l_2)$ is a bijection.

We first show that $\mathcal{F}$ is essentially surjective. Given a finite separable extension $l/k$, we may apply the primitive element theorem to write

$$l \simeq k(\bar{\alpha}) = \frac{k[x]}{(g(x))},$$

for some $\bar{\alpha} \in l$ whose minimal polynomial $\bar{g} \in k[x]$ is necessarily monic, irreducible, separable, and of degree $n := [l : k]$. Let $g \in A[x]$ be any monic lift of $\bar{g}$; then $g$ is also irreducible, separable, and of degree $n$. Now let

$$L := \frac{K[x]}{(g(x))} = K(\alpha),$$

where $\alpha$ is the image of $x$ in $K[x]/g(x)$. Then $L/K$ is a finite separable extension, and by Corollary 10.13, $(p, g(\alpha))$ is the unique maximal ideal of $A[\alpha]$ (since $\bar{g}$ is irreducible) and

$$\frac{B}{q} \simeq \frac{A[\alpha]}{(p, g(\alpha))} \simeq \frac{A[x]}{(p, g(x))} \simeq \frac{(A/p)[x]}{(\bar{g}(x))} \simeq l.$$

We thus have $[L : K] = \deg g = [l : k] = n$, and it follows that $L/K$ is an unramified extension of degree $n = f := [l : k]$: the ramification index of $q$ is necessarily $e = n/f = 1$, and the extension $l/k$ is separable by assumption (so in fact $B = A[\alpha]$, by Theorem 10.14).

We now show that the functor $\mathcal{F}$ is full and faithful. Given finite unramified extensions $L_1, L_2$ with valuation rings $B_1, B_2$ and residue fields $l_1, l_2$, we have induced maps

$$\text{Hom}_K(L_1, L_2) \overset{\sim}{\longrightarrow} \text{Hom}_A(B_1, B_2) \longrightarrow \text{Hom}_k(l_1, l_2).$$

The first map is given by restriction from $L_1$ to $B_1$, and since tensoring with $K$ gives an inverse map in the other direction, it is a bijection. We need to show that the same is true of the second map, which sends $\varphi : B_1 \to B_2$ to the $k$-homomorphism $\overline{\varphi}$ that sends $\overline{\alpha} \in l_1 = B_1/q_1$ to the reduction of $\varphi(\alpha)$ modulo $q_2$, where $\alpha$ is any lift of $\alpha$.

As above, use the primitive element theorem to write $l_1 = k(\bar{\alpha}) = k[x]/(\bar{g}(x))$ for some $\bar{\alpha} \in l_1$. If we now lift $\bar{\alpha}$ to $\alpha \in B_1$, we must have $L_1 = K(\alpha)$, since $[L_1 : K] = [l_1 : k]$ is equal to the degree of the minimal polynomial $\bar{g}$ of $\bar{\alpha}$ which cannot be less than the degree of the minimal polynomial $g$ of $\alpha$ (both are monic). Moreover, we also have $B_1 = A[\alpha]$, since this is true of the valuation ring of every finite unramified extension in our category.

Each $A$-module homomorphism in

$$\text{Hom}_A(B_1, B_2) = \text{Hom}_A\left(\frac{A[x]}{(g(x))}, B_2\right)$$

is uniquely determined by the image of $x$ in $B_2$. Thus gives us a bijection between $\text{Hom}_A(B_1, B_2)$ and the roots of $g$ in $B_2$. Similarly, each $k$-algebra homomorphism in

$$\text{Hom}_k(l_1, l_2) = \text{Hom}_k\left(\frac{k[x]}{(\overline{g}(x))}, l_2\right)$$

is uniquely determined by the image of $x$ in $l_2$, and there is a bijection between $\text{Hom}_k(l_1, l_2)$ and the roots of $\overline{g}$ in $l_2$. Now $\bar{g}$ is separable, so every root of $\bar{g}$ in $l_2 = B_2/q_2$ lifts to a unique root of $g$ in $B_2$, by Hensel’s Lemma 9.15. Thus the map $\text{Hom}_A(B_1, B_2) \longrightarrow \text{Hom}_k(l_1, l_2)$ induced by $\mathcal{F}$ is a bijection.

\[\square\]
Remark 10.16. In the proof above we actually only used the fact that \( L_1/K \) is unramified. The map \( \text{Hom}_K(L_1, L_2) \to \text{Hom}_k(l_1, l_2) \) is a bijection even if \( L_2/K \) is not unramified.

Let us note the following corollary, which follows from our proof of Theorem 10.15.

Corollary 10.17. Assume \( A \subset K \subset L \) with \( A \) a complete DVR with residue field \( k \). Then \( L/K \) is unramified if and only if \( B = A[\alpha] \) for some \( \alpha \in L \) whose minimal polynomial \( g \in A[x] \) has separable image \( \bar{g} \) in \( k[x] \).

Proof. The forward direction was proved in the proof of the theorem, and for the reverse direction note that \( \bar{g} \) must be irreducible, since otherwise we could use Hensel’s lemma to lift a non-trivial factorization of \( \bar{g} \) to a non-trivial factorization of \( g \), so the residue field extension is separable and has the same degree as \( L/K \), so \( L/K \) is unramified.

Corollary 10.18. Let \( A \) be a complete DVR with fraction field \( K \) and residue field \( k \), and let \( \zeta_n \) be a primitive \( n \)th root of unity in some algebraic closure of \( K \), with \( n \) prime to the characteristic of \( k \). The extension \( K(\zeta_n)/K \) is unramified.

Proof. The field \( K(\zeta_n) \) is the splitting field of \( f(x) = x^n - 1 \) over \( K \). The image \( \bar{f} \) of \( f \) in \( k[x] \) is separable when \( p \nmid n \), since \( \gcd(\bar{f}, \bar{f}') \neq 1 \) only when \( \bar{f}' = nx^{n-1} \) is zero, equivalently, only when \( p|n \). When \( \bar{f} \) is separable, so are all of its divisors, including the reduction of the minimal polynomial of \( \zeta_n \), which must be irreducible since otherwise we could obtain a contradiction by lifting a non-trivial factorization via Hensel’s lemma. It follows that the residue field of \( K(\zeta_n) \) is a separable extension of \( k \), thus \( K(\zeta_n)/K \) is unramified.

When the residue field \( k \) is finite (always the case if \( K \) is a local field), we can give a precise description of the finite unramified extensions \( L/K \).

Corollary 10.19. Let \( A \) be a complete DVR with fraction field \( K \) and finite residue field \( \mathbb{F}_q \). An extension \( L/K \) is unramified if and only if \( L \cong K(\zeta_{q^n-1}) \), where \( n := [L:K] \). When this holds, \( B \cong A[\zeta_{q^n-1}] \) is the integral closure of \( A \) in \( L \) and \( L/K \) is a Galois extension with \( \text{Gal}(L/K) \cong \mathbb{Z}/n\mathbb{Z} \).

Proof. By the previous corollary, \( L \cong K(\zeta_{q^n-1}) \) is unramified, and it has degree \( n \) because its residue field \( l \cong \mathbb{F}_{q^n} \) is the splitting field of \( x^{q^n-1} - 1 \) over \( \mathbb{F}_q \), which is an extension of degree \( n \) (indeed, one can take this as the definition of \( \mathbb{F}_{q^n} \)).

We now show that if \( L/K \) is unramified of degree \( n \), then \( L = K(\zeta_{q^n-1}) \). The residue field has degree \( n \) and is thus isomorphic to \( \mathbb{F}_{q^n} \), so its multiplicative group is a cyclic of order \( q^n - 1 \) generated by some \( \bar{\alpha} \). The minimal polynomial \( \bar{g} \in \mathbb{F}_q[x] \) of \( \bar{\alpha} \) divides \( x^{q^n-1} - 1 \), and since \( \bar{g} \) is irreducible, it is coprime to the quotient \( (x^{q^n-1} - 1)/\bar{g} \). By Hensel’s Lemma 9.19, we can lift \( \bar{g} \) to a polynomial \( g \in A[x] \) that divides \( x^{q^n-1} - 1 \in A[x] \), and by Hensel’s Lemma 9.15 we can lift \( \bar{\alpha} \) to a root \( \alpha \) of \( g \), in which case \( \alpha \) is also a root of \( x^{q^n-1} - 1 \); it must be a primitive \( (q^n - 1) \)-root of unity because its reduction \( \bar{\alpha} \) is.

We have \( B \cong A[\zeta_{q^n-1}] \) by Theorem 10.14, and \( L \) is the splitting field of \( x^{q^n-1} - 1 \), since its residue field \( \mathbb{F}_{q^n} \) is (we can lift the factorization of \( x^{q^n-1} - 1 \) from \( \mathbb{F}_{q^n} \) to \( L \) via Hensel’s lemma). It follows that \( L/K \) is Galois, and the bijection between \( (q^n - 1) \)-roots of unity in \( L \) and \( \mathbb{F}_{q^n} \) induces an isomorphism \( \text{Gal}(L/K) \cong \text{Gal}(l/k) = \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z} \).

Corollary 10.20. Let \( A \) be a complete DVR with fraction field \( K \) and finite residue field of characteristic \( p \), and suppose that \( K \) does not contain a primitive \( p \)th root of unity. The extension \( K(\zeta_m)/K \) is ramified if and only if \( p \) divides \( m \).
Proof. If \( p \) does not divide \( m \) then Corollary 10.18 implies that \( K(\zeta_m)/K \) is unramified. If \( p \) divides \( m \) then \( K(\zeta_m) \) contains \( K(\zeta_p) \), which by Corollary 10.19 is unramified if and only if \( K(\zeta_p) \simeq K(\zeta_{p^n-1}) \) with \( n := [K(\zeta_p) : K] \), which occurs if and only if \( p \) divides \( p^n - 1 \) (since \( \zeta_p \not\in K \)), which it does not; thus \( K(\zeta_p) \) and therefore \( K(\zeta_m) \) is ramified when \( p|m \). \( \square \)

Example 10.21. Consider \( A = \mathbb{Z}_p \), \( K = \mathbb{Q}_p \), \( k = \mathbb{F}_p \), and fix \( \overline{\mathbb{F}}_p \) and \( \overline{\mathbb{Q}}_p \). For each positive integer \( n \), the finite field \( \mathbb{F}_p \) has a unique extension of degree \( n \) in \( \overline{\mathbb{F}}_p \), namely, \( \mathbb{F}_p^n \). Thus for each positive integer \( n \), the local field \( \mathbb{Q}_p \) has a unique unramified extension of degree \( n \); it can be explicitly constructed by adjoining a primitive root of unity \( \zeta_{p^n-1} \) to \( \mathbb{Q}_p \). The element \( \zeta_{p^n-1} \) will necessarily have minimal polynomial of degree \( n \) dividing \( x^{p^n-1} - 1 \).

Another useful consequence of Theorem 10.15 that applies when the residue field is finite is that the norm map \( N_{L/K} \) restricts to a surjective map \( B^\times \to A^\times \) on unit groups; in fact, this property characterizes unramified extensions.

Theorem 10.22. Assume \( AKLB \) with \( A \) a complete DVR with finite residue field. Then \( L/K \) is unramified if and only if \( N_{L/K}(B^\times) = A^\times \).

Proof. See Problem Set 6. \( \square \)

Definition 10.23. Let \( L/K \) be a separable extension. The maximal unramified extension of \( K \) in \( L \) is the subfield

\[
\bigcup_{K \subseteq E \subseteq L \atop E/K \text{ fin. unram.}} E \subseteq L
\]

where the union is over finite unramified subextensions \( E/K \). When \( L = K^{sep} \) is the separable closure of \( K \), this is the maximal unramified extension of \( K \), denoted \( K^{unr} \).

Example 10.24. The field \( \mathbb{Q}_p^{unr} \) is an infinite extension of \( \mathbb{Q}_p \) with Galois group

\[
\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \lim_{\leftarrow n} \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p) \simeq \lim_{\leftarrow n} \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}},
\]

where the inverse limit is taken over positive integers \( n \) ordered by divisibility. The ring \( \hat{\mathbb{Z}} \) is the profinite completion of \( \mathbb{Z} \). The field \( \mathbb{Q}_p^{unr} \) has value group \( \mathbb{Z} \) and residue field \( \mathbb{F}_p \).

Theorem 10.25. Assume \( AKLB \) with \( A \) a complete DVR and separable residue field extension \( l/k \). Let \( e_{L/K} \) and \( f_{L/K} \) be the ramification index and residue field degrees, respectively. The following hold:

(i) There is a unique intermediate field extension \( E/K \) that contains every unramified extension of \( K \) in \( L \) and it has degree \( [E : K] = f_{L/K} \).

(ii) The extension \( L/E \) is totally ramified and has degree \( [L : E] = e_{L/K} \).

(iii) If \( L/K \) is Galois then \( \text{Gal}(L/E) = I_{L/K} \), where \( I_{L/K} = I_q \) is the inertia subgroup of \( \text{Gal}(L/K) \) for the unique prime \( q \) of \( B \).

Proof. (i) Let \( E/K \) be the finite unramified extension of \( K \) in \( L \) corresponding to the finite separable extension \( l/k \) given by the functor \( \mathcal{F} \) in Theorem 10.15; then \( [E : K] = [l : k] = f_{L/K} \) as desired. The image of the inclusion \( l \subseteq l \) of the residue fields of \( E \) and \( L \) induces a field embedding \( E \hookrightarrow L \) in \( \text{Hom}_K(E, L) \), via the functor \( \mathcal{F} \). Thus we may regard \( E \) as a subfield of \( L \), and it is unique up to isomorphism. If \( E'/K \) is any other unramified
extension of $K$ in $L$ with residue field $k'$, then the inclusions $k' \subseteq l \subseteq l$ induce embeddings $E' \subseteq E \subseteq L$ that must be inclusions.

(ii) We have $f_{L/E} = [l : l] = 1$, so $e_{L/E} = [L : E] = [L : K]/[E : K] = e_{L/K}$.

(iii) By Proposition 7.23, we have $I_{L/E} = \text{Gal}(L/E) \cap I_{L/K}$, and these three groups all have the same order $e_{L/K}$ so they must coincide.

\begin{flushright}
$\square$
\end{flushright}

References


