10 Extensions of complete DVRs

Recall that in our $AKLB$ setup, $A$ is a Dedekind domain with fraction field $K$, the field $L$ is a finite separable extension of $K$, and $B$ is the integral closure of $A$ in $L$; as we proved in Theorem 5.22, this implies that $B$ is also a Dedekind domain (with $L$ as its fraction field).

We now want to consider the special case where $A$ is a complete DVR; in this case $B$ is also a complete DVR, but this will take a little bit of work to prove. We first show that $B$ is a DVR.

**Theorem 10.1.** Assume $AKLB$ and that $A$ is a complete DVR with maximal ideal $p$. Then $B$ is a DVR whose maximal ideal $q$ is necessarily the unique prime above $p$.

**Proof.** We first show that $\#\{q|p}\} = 1$. At least one prime $q$ of $B$ lies above $p$, since the factorization of $qB \subset B$ is non-trivial. Now suppose for the sake of contradiction that $q_1, q_2 \in \{q|p}\}$ with $q_1 \neq q_2$. Choose $b \in q_1 - q_2$ and consider the ring $A[b] \subset B$. The ideals $q_1 \cap A[b]$ and $q_2 \cap A[b]$ are distinct prime ideals of $A[b]$ containing $pA[b]$, and both are maximal, since they are nonzero and $\dim A[b] = \dim A = 1$ (note that $A[b] \subset B$ is integral over $A$ and therefore has the same dimension). The quotient ring $A[b]/pA[b]$ thus has at least two maximal ideals. Let $f \in A[x]$ be the minimal polynomial of $b$ over $K$, and let $\bar{f} \in k[x]$ be its reduction to the residue field $A/p$. We have

$$\frac{(A/p)[x]}{(f)} \simeq \frac{A[x]}{(p, f)} \simeq \frac{A[b]}{pA[b]},$$

thus the ring $(A/p)[x]/(\bar{f})$ has at least two maximal ideals, which implies that $\bar{f}$ is divisible by two distinct irreducible polynomials (because $(A/p)[x]$ is a PID). We can thus factor $\bar{f} = gh$ with $g$ and $h$ coprime. By Hensel's Lemma 9.19, we can lift this to a non-trivial factorization $f = gh$ of $f$ in $A[x]$, contradicting the irreducibility of $f$.

Every maximal ideal of $B$ lies above a maximal ideal of $A$, but $A$ has only the maximal ideal $p$ and $\#\{q|p\} = 1$, so $B$ has a unique (nonzero) maximal ideal $q$. Thus $B$ is a local Dedekind domain, hence a local PID, and not a field, so $B$ is a DVR, by Theorem 1.15.

**Remark 10.2.** The assumption that $A$ is complete is necessary. For example, if $A$ is the DVR $\mathbb{Z}_5$ with fraction field $K = \mathbb{Q}$ and we take $L = \mathbb{Q}(i)$, then the integral closure of $A$ in $L$ is $B = \mathbb{Z}_5[i]$, which is a PID but not a DVR: the ideals $(1 + 2i)$ and $(1 - 2i)$ are both maximal (and not equal). But if we take completions we get $A = Z_5$ and $K = \mathbb{Q}_5$, and now $L = \mathbb{Q}_5(i) = \mathbb{Q}_5 = K$, since $x^2 + 1$ has a root in $\mathbb{F}_5 \simeq \mathbb{Z}_5/5\mathbb{Z}_5$ that we can lift to $\mathbb{Z}_5$ via Hensel's lemma; thus if we complete $A$ then $B = A$ is a DVR as required.

**Definition 10.3.** Let $K$ be a field with absolute value $| |$ and let $V$ be a $K$-vector space. A **norm** on $V$ is a function $\| \| : V \to \mathbb{R}_{\geq 0}$ such that

- $\|v\| = 0$ if and only if $v = 0$.
- $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in K$ and $v \in V$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Each norm $\| \|$ induces a topology on $V$ via the distance metric $d(v, w) := \|v - w\|$.

**Example 10.4.** Let $V$ be a $K$-vector space with basis $(e_i)$, and for $v \in V$ let $v_i \in K$ denote the coefficient of $e_i$ in $v = \sum_i v_ie_i$. The **sup-norm** $\|v\|_\infty := \sup \{|v_i|\}$ is a norm on $V$ (so...
every vector space has at least one norm). If $V$ is also a $K$-algebra, an absolute value $\| \|$ on $V$ (as a ring) is a norm on $V$ (as a $K$-vector space) if and only if it extends the absolute value on $K$ (fix $v \neq 0$ and note that $\| \lambda \| \| v \| = \| \lambda v \| = \| \lambda \| \| v \| \iff \| \lambda \| = | \lambda |$).

**Proposition 10.5.** Let $V$ be a vector space of finite dimension over a complete field $K$. Every norm on $V$ induces the same topology, in which $V$ is a complete metric space.

**Proof.** See Problem Set 5. \hfill \Box

**Theorem 10.6.** Let $A$ be a complete DVR with fraction field $K$, maximal ideal $\mathfrak{p}$, discrete valuation $v_\mathfrak{p}$, and absolute value $|x|_\mathfrak{p} := c^{v_\mathfrak{p}(x)}$, with $0 < c < 1$. Let $L/K$ be a finite extension of degree $n$. The following hold.

(i) There is a unique absolute value $|x| := |N_{L/K}(x)|_\mathfrak{p}^{1/n}$ on $L$ that extends $| \cdot |_\mathfrak{p}$;

(ii) The field $L$ is complete with respect to $| \cdot |$, and its valuation ring $\{ x \in L : |x| \leq 1 \}$ is equal to the integral closure $B$ of $A$ in $L$;

(iii) If $L/K$ is separable then $B$ is a complete DVR whose maximal ideal $\mathfrak{q}$ induces

$$|x| = |x|_\mathfrak{q} := c^{\frac{1}{e_\mathfrak{q}} v_\mathfrak{q}(x)},$$

where $e_\mathfrak{q}$ is the ramification index of $\mathfrak{q}$, that is, $\mathfrak{p}B = \mathfrak{q}^{e_\mathfrak{q}}$.

**Proof.** Assuming for the moment that $| \cdot |$ is actually an absolute value (which is not obvious!), for any $x \in K$ we have

$$|x| = |N_{L/K}(x)|_\mathfrak{p}^{1/n} = |x^n|_\mathfrak{p}^{1/n} = |x|_\mathfrak{p},$$

so $| \cdot |$ extends $| \cdot |_\mathfrak{p}$ and is therefore a norm on $L$. The fact that $| \cdot |_\mathfrak{p}$ is nontrivial means that $|x|_\mathfrak{p} \neq 1$ for some $x \in K^\times$, and $|x|^a = |x|_\mathfrak{p} = |x|$ only for $a = 1$, which implies that $| \cdot |$ is the unique absolute value in its equivalence class extending $| \cdot |_\mathfrak{p}$. Every norm on $L$ induces the same topology (by Proposition 10.5), so $| \cdot |$ is the only absolute value on $L$ that extends $| \cdot |_\mathfrak{p}$.

We now show $| \cdot |$ is an absolute value. Clearly $|x| = 0 \iff x = 0$ and $| \cdot |$ is multiplicative; we only need to check the triangle inequality. It suffices to show $|x| \leq 1 \Rightarrow |x+1| \leq |x|+1$, since we always have $|y+z| = |z| |y/z+1|$ and $|y+z| = |z| |(y/z)+1|$, and without loss of generality we assume $|y| \leq |z|$. In fact the stronger implication $|x| \leq 1 \Rightarrow |x+1| \leq 1$ holds:

$$|x| \leq 1 \iff |N_{L/K}(x)|_\mathfrak{p} \leq 1 \iff N_{L/K}(x) \in A \iff x \in B \iff x+1 \in B \iff |x+1| \leq 1.$$

The first biconditional follows from the definition of $| \cdot |$, the second follows from the definition of $| \cdot |_\mathfrak{p}$, the third is Corollary 9.21, the fourth is obvious, and the fifth follows from the first three after replacing $x$ with $x+1$. This completes the proof of (i), and also proves (ii).

We now assume $L/K$ is separable. Then $B$ is a DVR, by Theorem 10.1, and it is complete because it is the valuation ring of $L$. Let $\mathfrak{q}$ be the unique maximal ideal of $B$. The valuation $v_\mathfrak{q}$ extends $v_\mathfrak{p}$ with index $e_\mathfrak{q}$, by Theorem 8.20, so $v_\mathfrak{q}(x) = e_\mathfrak{q}v_\mathfrak{p}(x)$ for $x \in K^\times$. We have $0 < e_\mathfrak{q} < 1$, so $|x|_\mathfrak{q} := (e_\mathfrak{q})^{v_\mathfrak{q}(x)}$ is an absolute value on $L$ induced by $v_\mathfrak{q}$. To show it is equal to $| \cdot |$, it suffices to show that it extends $| \cdot |_\mathfrak{p}$, since we already know that $| \cdot |$ is the unique absolute value on $L$ with this property. For $x \in K^\times$ we have

$$|x|_\mathfrak{q} = c^{\frac{1}{e_\mathfrak{q}} v_\mathfrak{q}(x)} = c^{\frac{1}{e_\mathfrak{q}} e_\mathfrak{q} v_\mathfrak{p}(x)} = c^{v_\mathfrak{p}(x)} = |x|_\mathfrak{p},$$

and the theorem follows. \hfill \Box
Remark 10.7. The transitivity of $N_{L/K}$ in towers (Corollary 4.52) implies that we can uniquely extend the absolute value on the fraction field $K$ of a complete DVR to an algebraic closure $\overline{K}$. In fact, this is another form of Hensel’s lemma in the following sense: one can show that a (not necessarily discrete) valuation ring $A$ is Henselian if and only if the absolute value of its fraction field $K$ can be uniquely extended to $\overline{K}$; see [4, Theorem 6.6].

Corollary 10.8. Assume $AKLB$ and that $A$ is a complete DVR with maximal ideal $p$ and let $q|p$. Then $v_q(x) = \frac{1}{e_q} v_p(N_{L/K}(x))$ for all $x \in L$.

Proof. $v_p(N_{L/K}(x)) = v_p(N_{L/K}((x))) = v_p(N_{L/K}(q^{v_q(x)})) = v_p(p^{e_q f_q(x)}) = f_q v_q(x)$. □

Remark 10.9. One can generalize the notion of a discrete valuation to a valuation, a surjective homomorphism $v: K^\times \to \Gamma$, in which $\Gamma$ is a (totally) ordered abelian group and $v(x + y) \leq \max(v(x), v(y))$; we extend $v$ to $K$ by defining $v(0) = \infty$ to be strictly greater than any element of $\Gamma$. In the $AKLB$ setup with $A$ a complete DVR, one can then define a valuation $v(x) = \frac{1}{e_q} v_q(x)$ with image $\frac{1}{e_q} \mathbb{Z}$ that restricts to the discrete valuation $v_p$ on $K$. The valuation $v$ then extends to a valuation on $\overline{K}$ with $\Gamma = \mathbb{Q}$. Some texts take this approach, but we will generally stick with discrete valuations (so our absolute value on $L$ restricts to $K$, but our discrete valuations on $L$ do not restrict to discrete valuations on $K$, they extend them with index $e_q$).

Remark 10.10. Recall that a valuation ring is an integral domain $A$ with fraction field $K$ such that for every $x \in K^\times$ either $x \in A$ or $x^{-1} \in A$ (possibly both). As you will show on Problem Set 6, if $A$ is a valuation ring, then there exists a valuation $v: K \to \Gamma \cup \{\infty\}$ for some totally ordered abelian group $\Gamma$ such that $A = \{x \in K : v(x) \geq 0\}$ is the valuation ring of $K$ with respect to this valuation.

In our $AKLB$ setup, if $A$ is a complete DVR with maximal ideal $p$ then $B$ is a complete DVR with maximal ideal $q|p$ and the formula $[L : K] = \sum_{p|q} e_q f_q$ given by Theorem 5.32 has only one term $e_q f_q$. We now simplify matters even further by reducing to the two extreme cases $f_q = 1$ (a totally ramified extension) and $e_q = 1$ (an unramified extension, provided that the residue field extension is separable). 1

10.1 The Dedekind-Kummer theorem in a local setting

Recall that the Dedekind-Kummer theorem (Theorem 6.14) allows us to factor primes in our $AKLB$ setting by factoring polynomials over the residue field, provided that $B$ is monogenic (of the form $A[\alpha]$ for some $\alpha \in B$), or the prime of interest does not contain the conductor. We now show that in the special case where $A$ and $B$ are DVRs and the residue field extension is separable, $B$ is always monogenic; this holds, for example, whenever $K$ is a local field. To prove this, we first recall a form of Nakayama’s lemma.

Lemma 10.11 (NAKAYAMA’S LEMMA). Let $A$ be a local ring with maximal ideal $p$, and let $M$ be a finitely generated $A$-module. If the images of $x_1, \ldots, x_n \in M$ generate $M/pM$ as an $(A/p)$-vector space then $x_1, \ldots, x_n$ generate $M$ as an $A$-module.

Proof. See [1, Corollary 4.8b]. □

1Recall from Definition 5.34 that separability of the residue field extension is part of the definition of an unramified extension. If the residue field is perfect (as when $K$ is a local field, for example), the residue field extension is automatically separable, but in general it need not be, even when $L/K$ is unramified.
Before proving our theorem on local monogenicity, we record a few corollaries of Nakayama’s Lemma that will be useful later.

**Corollary 10.12.** Let $A$ be a local noetherian ring with maximal ideal $p$, let $g \in A[x]$, and let $B := A[x]/(g(x))$. Every maximal ideal $m$ of $B$ contains the ideal $pB$.

**Proof.** Suppose not. Then $m + pB = B$ for some maximal ideal $m$ of $B$. The ring $B$ is finitely generated over the noetherian ring $A$, hence a noetherian $A$-module, so its $A$-submodules are all finitely generated. Let $z_1, \ldots, z_n$ be $A$-module generators for $m$. Every coset of $pB$ in $B$ can be written as $z + pB$ for some $A$-linear combination $z$ of $z_1, \ldots, z_n$, so the images of $z_1, \ldots, z_n$ generate $B/pB$ as an $(A/p)$-vector space. By Nakayama’s lemma, $z_1, \ldots, z_n$ generate $B$, in which case $m = B$, a contradiction. □

As a corollary, we immediately obtain a local version of the Dedekind-Kummer theorem that does not even require $A$ and $B$ to be Dedekind domains.

**Corollary 10.13.** Let $A$ be a local noetherian ring with maximal ideal $p$, let $g \in A[x]$ be a polynomial with reduction $\bar{g} \in (A/p)[x]$, and let $\alpha$ be the image of $x$ in the ring $B := A[x]/(g(x)) = A[\alpha]$. The maximal ideals of $B$ are $(p, g_i(\alpha))$, where $g_1, \ldots, g_m \in A[x]$ are lifts of the distinct irreducible polynomials $\bar{g}_i \in (A/p)[x]$ that divide $\bar{g}$.

**Proof.** By Corollary 10.12, the quotient map $B \to B/pB$ gives a one-to-one correspondence between maximal ideals of $B$ and maximal ideals of $B/pB$, and we have

$$\frac{B}{pB} \cong \frac{A[x]}{(p, g(x))} \cong \frac{(A/p)[x]}{(\bar{g}(x))}.$$  

Each maximal ideal of $(A/p)[x]/(\bar{g}(x))$ is the reduction of an irreducible divisor of $\bar{g}$, hence one of the $\bar{g}_i$ (because $(A/p)[x]$ is a PID). The corollary follows. □

**Theorem 10.14.** Assume $AKLB$, with $A$ and $B$ DVRs with residue fields $k := A/p$ and $l := B/q$. If $l/k$ is separable then $B = A[\alpha]$ for some $\alpha \in B$; if $L/K$ is unramified this holds for any $\alpha \in B$ whose image generates the residue field extension $l/k$.

**Proof.** Let $pB = q^e$ be the factorization of $pB$ and let $f = [l : k]$ be the residue field degree, so that $ef = n := [L : K]$. The extension $l/k$ is separable, so we may apply the primitive element theorem to write $l = k(\alpha_0)$ for some $\alpha_0 \in l$ whose minimal polynomial $\bar{g}$ is separable of degree equal to $f$. Let $g \in A[x]$ be a monic lift of $\bar{g}$, and let $\alpha_0$ be any lift of $\bar{\alpha}_0$ to $B$. It $v_q(g(\alpha_0)) = 1$ then let $\alpha := \alpha_0$. Otherwise, let $\pi_0$ be any uniformizer for $B$ and let $\alpha := \alpha_0 + \pi_0 \in B$ (so $\alpha \equiv \bar{\alpha}_0 \mod q$) Writing $g(x + \pi_0) = g(x) + \pi_0 g'(x) + \pi_0^2 h(x)$ for some $h \in A[x]$ via Lemma 9.11, we have

$$v_q(g(\alpha)) = v_q(g(\alpha_0 + \pi_0)) = v_q(g(\alpha_0) + \pi_0 g'(\alpha_0) + \pi_0^2 h(\alpha_0)) = 1,$$

so $\pi := g(\alpha)$ is also a uniformizer for $B$.

We now claim $B = A[\alpha]$, equivalently, that $1, \alpha, \ldots, \alpha^{n-1}$ generate $B$ as an $A$-module. By Nakayama’s lemma, it suffices to show that the reductions of $1, \alpha, \ldots, \alpha^{n-1}$ span $B/pB$ as an $k$-vector space. We have $p = q^e$, so $pB = (\pi^e)$. We can represent each element of $B/pB$ as a coset

$$b + pB = b_0 + b_1 \pi + b_2 \pi^2 \cdots + b_{e-1} \pi^{e-1} + pB,$$
where \(b_0, \ldots, b_{e-1}\) are determined up to equivalence modulo \(\pi B\). Now \(1, \bar{\alpha}, \ldots, \bar{\alpha}^{f-1}\) are a basis for \(B/\pi B = B/q\) as a \(k\)-vector space, and \(\pi = g(\alpha)\), so we can rewrite this as
\[
\begin{align*}
    b + pB &= (a_0 + a_1\alpha + \cdots + a_{f-1}\alpha^{f-1}) + (a_f + a_{f+1}\alpha + \cdots + a_{2f-1}\alpha^{f-1})g(\alpha) + \cdots \\
    &\quad + (a_{ef-f+1} + a_{ef-f+2}\alpha + \cdots + a_{ef-1}\alpha^{f-1})g(\alpha)^{e-1} + pB.
\end{align*}
\]
Since \(\deg g = f\), and \(n = ef\), this expresses \(b + pB\) in the form \(b' + pB\) with \(b'\) in the \(A\)-span of \(1, \ldots, \alpha^{n-1}\). Thus \(B = A[\alpha]\).

We now note that if \(L/K\) is unramified then \(l/k\) is separable (this is part of the definition of unramified), and \(e = 1\), \(f = n\), in which case there is no need to require \(g(\alpha)\) to be a uniformizer and we can just take \(\alpha = \alpha_0\) to be any lift of any \(\bar{\alpha}_0\) that generates \(l\) over \(k\). \(\square\)

10.2 Unramified extensions of a complete DVR

Let \(A\) be a complete DVR with fraction field \(K\) and residue field \(k\). Associated to any finite unramified extension of \(L/K\) of degree \(n\) is a corresponding finite separable extension of residue fields \(l/k\) of the same degree \(n\). Given that the extensions \(L/K\) and \(l/k\) are finite separable extensions of the same degree, we might wonder how they are related. More precisely, if we fix \(K\) with residue field \(k\), what is the relationship between finite unramified extensions \(L/K\) of degree \(n\) and finite separable extensions \(l/k\) of degree \(n^2\)? Each \(L/K\) uniquely determines a corresponding \(l/k\), but what about the converse?

This question has a surprisingly nice answer. The finite unramified extensions \(L\) of \(K\) form a category \(C^\text{unr}_K\) whose morphisms are \(K\)-algebra homomorphisms, and the finite separable extensions \(l\) of \(k\) form a category \(C^\text{sep}_k\) whose morphisms are \(k\)-algebra homomorphisms. These two categories are equivalent.

**Theorem 10.15.** Let \(A\) be a complete DVR with fraction field \(K\) and residue field \(k := A/p\). The categories \(C^\text{unr}_K\) and \(C^\text{sep}_k\) are equivalent via the functor \(\mathcal{F}: C^\text{unr}_K \to C^\text{sep}_k\) that sends each unramified extension \(\bar{L}\) of \(K\) to its residue field \(l\), and each \(K\)-algebra homomorphism \(\varphi: L_1 \to L_2\) to the \(k\)-algebra homomorphism \(\bar{\varphi}: l_1 \to l_2\) defined by \(\bar{\varphi}(\bar{\alpha}) := \bar{\varphi}(\bar{\alpha})\), where \(\alpha\) is any lift of \(\bar{\alpha} \in l_1 := B_1/q_1\) to \(B_1\) and \(\bar{\varphi}(\alpha)\) is the reduction of \(\varphi(\alpha) \in B_2\) to \(l_2 := B_2/q_2\); here \(q_1, q_2\) are the maximal ideals of the valuation rings \(B_1, B_2\) of \(L_1, L_2\), respectively.

In particular, \(\mathcal{F}\) gives a bijection between the isomorphism classes in \(C^\text{unr}_K\) and \(C^\text{sep}_k\), and if \(L_1, L_2\) and have residue fields \(l_1, l_2\) then \(\mathcal{F}\) induces a bijection of finite sets
\[
\text{Hom}_K(L_1, L_2) \sim \text{Hom}_k(l_1, l_2).
\]

**Proof.** Let us first verify that \(\mathcal{F}\) is well-defined. It is clear that it maps finite unramified extensions \(L/K\) to finite separable extension \(l/k\), but we should check that the map on morphisms does not depend on the lift \(\alpha\) of \(\bar{\alpha}\) we pick. So let \(\varphi: L_1 \to L_2\) be a \(K\)-algebra homomorphism, and for \(\bar{\alpha} \in l_1\), let \(\alpha\) and \(\alpha'\) be two lifts of \(\bar{\alpha}\) to \(B_1\). Then \(\alpha - \alpha' \in q_1\), and this implies that \(\varphi(\alpha - \alpha') \in \varphi(q_1) = \varphi(B_1) \cap q_2 \subseteq q_2\), and therefore \(\bar{\varphi}(\alpha) = \bar{\varphi}(\alpha')\). The identity \(\varphi(q_1) = \varphi(B_1) \cap q_2 \subseteq q_2\) follows from the fact that \(\varphi\) restricts to an injective ring homomorphism \(B_1 \to B_2\) and \(B_2/\varphi(B_1)\) is a finite extension of DVRs in which \(q_2\) lies over the prime \(\varphi(q_1)\) of \(\varphi(B_1)\). It’s easy to see that \(\mathcal{F}\) sends identity morphisms to identity morphisms and that it is compatible with composition, so we have a well-defined functor.

To show that \(\mathcal{F}\) is an equivalence of categories we need to prove two things:
• \( \mathcal{F} \) is essentially surjective: each separable \( l/k \) is isomorphic to the residue field of some unramified \( L/K \)

• \( \mathcal{F} \) is full and faithful: the induced map \( \text{Hom}_K(L_1, L_2) \to \text{Hom}_k(l_1, l_2) \) is a bijection.

We first show that \( \mathcal{F} \) is essentially surjective. Given a finite separable extension \( l/k \), we may apply the primitive element theorem to write

\[
l \simeq k(\bar{\alpha}) = \frac{k[x]}{(\bar{g}(x))},
\]

for some \( \bar{\alpha} \in l \) whose minimal polynomial \( \bar{g} \in k[x] \) is necessarily monic, irreducible, separable, and of degree \( n := [l : k] \). Let \( g \in A[x] \) be any monic lift of \( \bar{g} \); then \( g \) is also irreducible, separable, and of degree \( n \). Now let

\[
L := \frac{K[x]}{(g(x))} = K(\alpha),
\]

where \( \alpha \) is the image of \( x \) in \( K[x]/g(x) \). Then \( L/K \) is a finite separable extension, and by Corollary 10.13, \((p, g(\alpha))\) is the unique maximal ideal of \( A[\alpha] \) (since \( \bar{g} \) is irreducible) and

\[
\frac{B}{q} \simeq \frac{A[\alpha]}{(p, g(\alpha))} \simeq \frac{A[x]}{(p, g(x))} \simeq \frac{(A/p)[x]}{(\bar{g}(x))} \simeq l.
\]

We thus have \([L : K] = \deg g = [l : k] = n\), and it follows that \( L/K \) is an unramified extension of degree \( n = f := [l : k] \); the ramification index of \( q \) is necessarily \( e = n/f = 1 \), and the extension \( l/k \) is separable by assumption (so in fact \( B = A[\alpha] \), by Theorem 10.14).

We now show that the functor \( \mathcal{F} \) is full and faithful. Given finite unramified extensions \( L_1, L_2 \) with valuation rings \( B_1, B_2 \) and residue fields \( l_1, l_2 \), we have induced maps

\[
\text{Hom}_K(L_1, L_2) \longrightarrow \text{Hom}_A(B_1, B_2) \longrightarrow \text{Hom}_k(l_1, l_2).
\]

The first map is given by restriction from \( L_1 \) to \( B_1 \), and since tensoring with \( K \) gives an inverse map in the other direction, it is a bijection. We need to show that the same is true of the second map, which sends \( \varphi : B_1 \to B_2 \) to the \( k \)-homomorphism \( \overline{\varphi} \) that sends \( \overline{\alpha} \in l_1 = B_1/q_1 \) to the reduction of \( \varphi(\alpha) \) modulo \( q_2 \), where \( \alpha \) is any lift of \( \overline{\alpha} \).

As above, use the primitive element theorem to write \( l_1 = k(\overline{\alpha}) = k[x]/(\bar{g}(x)) \) for some \( \overline{\alpha} \in l_1 \). If we now lift \( \overline{\alpha} \) to \( \alpha \in B_1 \), we must have \( L_1 = K(\alpha) \), since \([L_1 : K] = [l_1 : k]\) is equal to the degree of the minimal polynomial \( \bar{g} \) of \( \overline{\alpha} \) which cannot be less than the degree of the minimal polynomial \( g \) of \( \alpha \) (both are monic). Moreover, we also have \( B_1 = A[\alpha] \), since this is true of the valuation ring of every finite unramified extension in our category.

Each \( A \)-module homomorphism in

\[
\text{Hom}_A(B_1, B_2) = \text{Hom}_A\left(\frac{A[x]}{(g(x))}, B_2\right)
\]

is uniquely determined by the image of \( x \) in \( B_2 \). Thus gives us a bijection between \( \text{Hom}_A(B_1, B_2) \) and the roots of \( g \) in \( B_2 \). Similarly, each \( k \)-algebra homomorphism in

\[
\text{Hom}_k(l_1, l_2) = \text{Hom}_k\left(\frac{k[x]}{(\bar{g}(x))}, l_2\right)
\]

is uniquely determined by the image of \( x \) in \( l_2 \), and there is a bijection between \( \text{Hom}_k(l_1, l_2) \) and the roots of \( \bar{g} \) in \( l_2 \). Now \( \bar{g} \) is separable, so every root of \( \bar{g} \) in \( l_2 = B_2/q_2 \) lifts to a unique root of \( g \) in \( B_2 \), by Hensel’s Lemma 9.15. Thus the map \( \text{Hom}_A(B_1, B_2) \longrightarrow \text{Hom}_k(l_1, l_2) \) induced by \( \mathcal{F} \) is a bijection.
Remark 10.16. In the proof above we actually only used the fact that $L_1/K$ is unramified. The map $\text{Hom}_K(L_1, L_2) \to \text{Hom}_k(l_1, l_2)$ is a bijection even if $L_2/K$ is not unramified.

Let us note the following corollary, which follows from our proof of Theorem 10.15.

Corollary 10.17. Assume $A \cap L B$ with $A$ a complete DVR with residue field $k$. Then $L/K$ is unramified if and only if $B = A[\alpha]$ for some $\alpha \in L$ whose minimal polynomial $g \in A[x]$ has separable image $\bar{g}$ in $k[x]$.

Proof. The forward direction was proved in the proof of the theorem, and for the reverse direction note that $\bar{g}$ must be irreducible, since otherwise we could use Hensel’s lemma to lift a non-trivial factorization of $\bar{g}$ to a non-trivial factorization of $g$, so the residue field extension is separable and has the same degree as $L/K$, so $L/K$ is unramified.

Corollary 10.18. Let $A$ be a complete DVR with fraction field $K$ and residue field $k$, and let $\zeta_n$ be a primitive $n$th root of unity in some algebraic closure of $\overline{K}$, with $n$ prime to the characteristic of $k$. The extension $K(\zeta_n)/K$ is unramified.

Proof. The field $K(\zeta_n)$ is the splitting field of $f(x) = x^n - 1$ over $K$. The image $\bar{f}$ of $f$ in $k[x]$ is separable when $p \nmid n$, since $\gcd(f, \bar{f}) \neq 1$ only when $\bar{f}' = nx^{n-1}$ is zero, equivalently, only when $p|n$. When $\bar{f}$ is separable, so are all of its divisors, including the reduction of the minimal polynomial of $\zeta_n$, which must be irreducible since otherwise we could obtain a contradiction by lifting a non-trivial factorization via Hensel’s lemma. It follows that the residue field of $K(\zeta_n)$ is a separable extension of $k$, thus $K(\zeta_n)/K$ is unramified.

When the residue field $k$ is finite (always the case if $K$ is a local field), we can give a precise description of the finite unramified extensions $L/K$.

Corollary 10.19. Let $A$ be a complete DVR with fraction field $K$ and finite residue field $\mathbb{F}_q$. An extension $L/K$ is unramified if and only if $L \simeq K(\zeta_{q^n-1})$, where $n := [L : K]$. When this holds, $B \simeq A[\zeta_{q^n-1}]$ is the integral closure of $A$ in $L$ and $L/K$ is a Galois extension with $\text{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}$.

Proof. By the previous corollary, $L \simeq K(\zeta_{q^n-1})$ is unramified, and it has degree $n$ because its residue field $l \simeq \mathbb{F}_{q^n}$ is the splitting field of $x^{q^n-1} - 1$ over $\mathbb{F}_q$, which is an extension of degree $n$ (indeed, one can take this as the definition of $\mathbb{F}_{q^n}$).

We now show that if $L/K$ is unramified of degree $n$, then $L = K(\zeta_{q^n-1})$. The residue field has degree $n$ and is thus isomorphic to $\mathbb{F}_{q^n}$, so its multiplicative group is a cyclic group of order $q^n - 1$ generated by some $\bar{\alpha}$. The minimal polynomial $\bar{g} \in \mathbb{F}_q[x]$ of $\bar{\alpha}$ divides $x^{q^n-1} - 1$, and since $\bar{g}$ is irreducible, it is coprime to the quotient $(x^{q^n-1} - 1)/\bar{g}$. By Hensel’s Lemma 9.19, we can lift $\bar{g}$ to a polynomial $g \in A[x]$ that divides $x^{q^n-1} - 1 \in A[x]$, and by Hensel’s Lemma 9.15 we can lift $\bar{\alpha}$ to a root $\alpha$ of $g$, in which case $\alpha$ is also a root of $x^{q^n-1} - 1$; it must be a primitive $(q^n - 1)$-root of unity because its reduction $\bar{\alpha}$ is.

We have $B \simeq A[\zeta_{q^n-1}]$ by Theorem 10.14, and $L$ is the splitting field of $x^{q^n-1} - 1$, since its residue field $\mathbb{F}_{q^n}$ is (we can lift the factorization of $x^{q^n-1} - 1$ from $\mathbb{F}_{q^n}$ to $L$ via Hensel’s lemma). It follows that $L/K$ is Galois, and the bijection between $(q^n - 1)$-roots of unity in $L$ and $\mathbb{F}_{q^n}$ induces an isomorphism $\text{Gal}(L/K) \simeq \text{Gal}(l/k) = \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \simeq \mathbb{Z}/n\mathbb{Z}$. □

Corollary 10.20. Let $A$ be a complete DVR with fraction field $K$ and finite residue field of characteristic $p$, and suppose that $K$ does not contain a primitive $p$th root of unity. The extension $K(\zeta_m)/K$ is ramified if and only if $p$ divides $m$. □
Proof. If $p$ does not divide $m$ then Corollary 10.18 implies that $K(\zeta_m)/K$ is unramified. If $p$ divides $m$ then $K(\zeta_m)$ contains $K(\zeta_p)$, which by Corollary 10.19 is unramified if and only if $K(\zeta_p) \simeq K(\zeta_{p^n-1})$ with $n := [K(\zeta_p) : K]$, which occurs if and only if $p$ divides $p^n - 1$ (since $\zeta_p \notin K$), which it does not; thus $K(\zeta_p)$ and therefore $K(\zeta_m)$ is ramified when $p|m$. \qed

Example 10.21. Consider $A = \mathbb{Z}_p$, $K = \mathbb{Q}_p$, $k = \mathbb{F}_p$, and fix $\overline{\mathbb{F}}_p$ and $\overline{\mathbb{Q}}_p$. For each positive integer $n$, the finite field $\mathbb{F}_p$ has a unique extension of degree $n$ in $\overline{\mathbb{F}}_p$, namely, $\mathbb{F}_p^n$. Thus for each positive integer $n$, the local field $\mathbb{Q}_p$ has a unique unramified extension of degree $n$; it can be explicitly constructed by adjoining a primitive root of unity $\zeta_{p^n-1}$ to $\mathbb{Q}_p$. The element $\zeta_{p^n-1}$ will necessarily have minimal polynomial of degree $n$ dividing $x^{p^n-1} - 1$.

Another useful consequence of Theorem 10.15 that applies when the residue field is finite is that the norm map $N_{L/K}$ restricts to a surjective map $B^\times \to A^\times$ on unit groups; in fact, this property characterizes unramified extensions.

Theorem 10.22. Assume $AKLB$ with $A$ a complete DVR with finite residue field. Then $L/K$ is unramified if and only if $N_{L/K}(B^\times) = A^\times$.

Proof. See Problem Set 6. \qed

Definition 10.23. Let $L/K$ be a separable extension. The maximal unramified extension of $K$ in $L$ is the subfield

$$\bigcup_{K \subseteq E \subseteq L, E/K \text{ fin. unram.}} E \subseteq L$$

where the union is over finite unramified subextensions $E/K$. When $L = K^{\text{sep}}$ is the separable closure of $K$, this is the maximal unramified extension of $K$, denoted $K^{\text{unr}}$.

Example 10.24. The field $\mathbb{Q}_p^{\text{unr}}$ is an infinite extension of $\mathbb{Q}_p$ with Galois group

$$\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \varprojlim_n \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p) \simeq \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}},$$

where the inverse limit is taken over positive integers $n$ ordered by divisibility. The ring $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. The field $\mathbb{Q}_p^{\text{unr}}$ has value group $\mathbb{Z}$ and residue field $\mathbb{F}_p$.

Theorem 10.25. Assume $AKLB$ with $A$ a complete DVR and separable residue field extension $l/k$. Let $e_{L/K}$ and $f_{L/K}$ be the ramification index and residue field degrees, respectively. The following hold:

(i) There is a unique intermediate field extension $E/K$ that contains every unramified extension of $K$ in $L$ and it has degree $[E : K] = f_{L/K}$.

(ii) The extension $L/E$ is totally ramified and has degree $[L : E] = e_{L/K}$.

(iii) If $L/K$ is Galois then $\text{Gal}(L/E) = I_{L/K}$, where $I_{L/K} = I_q$ is the inertia subgroup of $\text{Gal}(L/K)$ for the unique prime $q$ of $B$.

Proof. (i) Let $E/K$ be the finite unramified extension of $K$ in $L$ corresponding to the finite separable extension $l/k$ given by the functor $\mathcal{F}$ in Theorem 10.15; then $[E : K] = [l : k] = f_{L/K}$ as desired. The image of the inclusion $l \subseteq l$ of the residue fields of $E$ and $L$ induces a field embedding $E \hookrightarrow L$ in $\text{Hom}_K(E, L)$, via the functor $\mathcal{F}$. Thus we may regard $E$ as a subfield of $L$, and it is unique up to isomorphism. If $E'/K$ is any other unramified
extension of $K$ in $L$ with residue field $k'$, then the inclusions $k' \subseteq l \subseteq l$ induce embeddings $E' \subseteq E \subseteq L$ that must be inclusions.

(ii) We have $f_{L/E} = [l : l] = 1$, so $e_{L/E} = [L : E] = [L : K]/[E : K] = e_{L/K}$.

(iii) By Proposition 7.23, we have $I_{L/E} = \text{Gal}(L/E) \cap I_{L/K}$; and these three groups all have the same order $e_{L/K}$ so they must coincide.

\[ \square \]

References


