# 25 Local class field theory

In this lecture we give a brief overview of local class field theory, following [1, Ch. 1]. Recall that a local field is a locally compact field whose topology is induced by a nontrivial absolute value (Definition 9.3). As we proved in Theorem 9.10, every local field is isomorphic to one of the following:

- $\mathbb{R}$ or $\mathbb{C}$ (archimedean, characteristic 0);
- finite extension of $\mathbb{Q}_p$ (nonarchimedean, characteristic 0);
- finite extension of $\mathbb{F}_q((t))$ (nonarchimedean, characteristic $p > 0$).

In the nonarchimedean cases, the ring of integers of a local field is a complete DVR with finite residue field.

The goal of local class field theory is to classify all finite abelian extensions of a given local field $K$. Rather than considering each finite abelian extension $L/K$ individually, we will treat them all at once, by fixing once and for all a separable closure $K^{\mathrm{sep}}$ of $K$ and working in the maximal abelian extension of $K$ inside $K^{\mathrm{sep}}$.

**Definition 25.1.** Let $K$ be field with separable closure $K^{\mathrm{sep}}$. The field

$$K^{\mathrm{ab}} := \bigcup_{\substack{L \subseteq K^{\mathrm{sep}} \\ L/K \text{ finite abelian}}} L$$

is the *maximal abelian extension of* $K$ (in $K^{\mathrm{sep}}$).

The field $K^{\mathrm{ab}}$ contains the $K^{\mathrm{unr}}$, the maximal unramified extension of $K$ in $K^{\mathrm{sep}}$; this is obvious in the archimedean case, and in the nonarchimedean case it follows from Theorem 10.16, which implies that $K^{\mathrm{unr}}$ is isomorphic to the algebraic closure of the residue field, which is abelian because it is pro-cyclic (every finite extension of the residue field is cyclic because the residue field is finite). We thus have a tower of field extensions

$$K \subseteq K^{\mathrm{unr}} \subseteq K^{\mathrm{ab}} \subseteq K^{\mathrm{sep}}.$$

The Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the profinite group

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \varprojlim_{L} \mathrm{Gal}(L/K),$$

where $L$ ranges over the finite abelian extensions of $K$ in $K^{\mathrm{sep}}$, ordered by inclusion. As a profinite group (like all Galois groups) $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is totally disconnected, compact, and Hausdorff (see Problem Set 11), and we have the Galois correspondence

$$\begin{aligned}
\{\text{extensions of } K \text{ in } K^{\mathrm{ab}}\} &\longleftrightarrow \{\text{ closed subgroups of } \mathrm{Gal}(K^{\mathrm{ab}}/K)\} \\
L &\longmapsto \mathrm{Gal}(K^{\mathrm{ab}}/L) \\
(K^{\mathrm{ab}})^H &\longleftarrow\!\shortmid \ H,
\end{aligned}$$

in which abelian extensions $L/K$ correspond to finite index open subgroups of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$; note that since $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is abelian, every subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is normal and it follows that every subextension of $K^{\mathrm{ab}}/K$ is Galois (and abelian).

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When $K$ is an archimedean local field its abelian extensions are easy to understand; either $K = \mathbb{R}$, in which case $\mathbb{C}$ is its only nontrivial abelian extension, or $K = \mathbb{C}$ and there are no nontrivial abelian extensions. Now suppose $K$ is a nonarchimedean local field with ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{p}$, and residue field $\mathbb{F}_\mathfrak{p} := \mathcal{O}_K/\mathfrak{p}$. If $L/K$ is a finite unramified extension with residue field $\mathbb{F}_\mathfrak{q} := \mathcal{O}_L/\mathfrak{q}$, Theorem 10.16 gives us a canonical isomorphism

$$\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p}) = \langle x \mapsto x^{\#\mathbb{F}_\mathfrak{p}} \rangle,$$

between the Galois group of $L/K$ and the Galois group of the residue field extension $\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p}$. The group $\mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p})$ is generated by the Frobenius automorphism $x \to x^{\#\mathbb{F}_\mathfrak{p}}$, and we use $\mathrm{Frob}_{L/K} \in \mathrm{Gal}(L/K)$ to denote the corresponding element of $\mathrm{Gal}(L/K)$. Thus each finite unramified extension of local fields $L/K$ comes equipped with a canonical generator $\mathrm{Frob}_{L/K}$ for its Galois group (and is necessarily cyclic).

## 25.1 Local Artin reciprocity

Local class field theory is based on the existence of a continuous homomorphism

$$\theta_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

known as the *local Artin homomorphism* (or *local reciprocity map*), which is characterized by the following theorem.

**Theorem 25.2** (Local Artin Reciprocity)**.** *Let $K$ be a local field. There is a unique continuous homomorphism*

$$\theta_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

*with the property that for each finite extension $L/K$ in $K^{\mathrm{ab}}$, the homomorphism*

$$\theta_{L/K} \colon K^\times \to \mathrm{Gal}(L/K)$$

*given by composing $\theta_K$ with the map $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(L/K)$ induced by restriction satisfies:*

- *if $K$ is nonarchimedean and $L/K$ is unramified then $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for every uniformizer $\pi$ of $\mathcal{O}_K$;*
- *$\theta_{L/K}$ is surjective with kernel $\mathrm{N}_{L/K}(L^\times)$, inducing $K^\times/\mathrm{N}_{L/K}(L^\times) \simeq \mathrm{Gal}(L/K)$.*

We will not prove this theorem in this course, but we would like to understand what it says (it says a lot). We first note that the homomorphisms $\theta_L/K$ form a compatible system, in the sense that if $L_1 \subseteq L_2$ then $\theta_{L_1/K}$ is the composition of $\theta_{L_2/K}$ with the map $\mathrm{Gal}(L_2/K) \to \mathrm{Gal}(L_1/K)$ induced by restriction. This follows from the fact that the maps induced by restriction are precisely the maps that define the inverse system $\varprojlim_L \mathrm{Gal}(L/K) \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K)$ that determines the structure of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ as a profinite group. Of course we can also view the map $\mathrm{Gal}(L_2/K) \to \mathrm{Gal}(L_1/K)$ as the quotient map $\mathrm{Gal}(L_2/K) \twoheadrightarrow \mathrm{Gal}(L_2/K)/\mathrm{Gal}(L_2/L_1)$.

It is first worth contrasting local Artin reciprocity with the more complicated global version of Artin reciprocity that we saw in Lecture 21:

- there is no modulus $\mathfrak{m}$ to worry about (not even a power of $\mathfrak{p}$); working in $K^{\mathrm{ab}}$ lets us treat all abelian extensions of $K$ in one fell swoop;
- there are no ray class groups $\mathrm{Cl}_K^\mathfrak{m}$ involved (the class group of a local field extension is always trivial); we instead consider quotients quotients of $K^\times$;
- the Takagi group $\mathcal{R}_K^\mathfrak{m}\mathrm{N}_{L/K}(\mathcal{I}_K^\mathfrak{m}) \subseteq \mathcal{I}_K^\mathfrak{m}$ is replaced by the *norm group* $\mathrm{N}_{L/K}(L^\times) \subseteq K^\times$.

## 25.2 Norm groups

**Definition 25.3.** Let $K$ be a local field. A *norm group* (of $K$) is a subgroup of the form

$$\mathrm{N}(L^\times) := \mathrm{N}_{L/K}(L^\times) \subseteq K^\times,$$

for some finite abelian extension $L/K$.

**Remark 25.4.** In fact, if $L/K$ is any finite extension (not necessarily abelian, not necessarily Galois), then $\mathrm{N}(L^\times) = \mathrm{N}(F^\times)$, where $F$ is the maximal abelian extension of $K$ in $L$; this is the Norm Limitation Theorem (see [1, Theorem III.3.5]). So we could have defined norm groups more generally. This is not relevant to classifying the abelian extension of $K$, but it demonstrates a key limitation of local class field theory (which extends to global class field theory): subgroups of $K^\times$ contain no information about nonabelian extensions of $K$.

Local Artin reciprocity implies that the Galois group of any finite abelian extension $L/K$ of a local fields is canonically isomorphic to the quotient $K^\times/\mathrm{N}_{L/K}(L^\times)$; thus to order to understand the finite abelian extensions of a local field $K$, we just need to understand its norm groups. In fact, Theorem 25.2 already tells us a quite a lot: in particular, the isomorphism $K^\times/\mathrm{N}(L^\times) \simeq \mathrm{Gal}(L/K)$ implies that $[K^\times : \mathrm{N}(L^\times)] = [L:K]$ is finite. Moreover, there is an order-reversing isomorphism between the lattice of norm groups in $K^\times$ and the lattice of finite abelian extensions of $K$; this is essentially the Galois correspondence with Galois groups replaced by norm groups.

**Corollary 25.5.** *The map $L \mapsto \mathrm{N}(L^\times)$ defines an inclusion reversing bijection between the finite abelian extensions $L/K$ in $K^{\mathrm{ab}}$ and the norm groups in $K^\times$ which satisfies*

$$(a)\ \mathrm{N}((L_1L_2)^\times) = \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) \qquad \textit{and} \qquad (b)\ \mathrm{N}((L_1 \cap L_2)^\times) = \mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times).$$

*Moreover, every subgroup of $K^\times$ that contains a norm group is a norm group.*

Here we write $L_1L_2$ for the compositum of $L_1$ and $L_2$ inside $K^{\mathrm{ab}}$ (the intersection of all subfields of $K^{\mathrm{ab}}$ that contain both $L_1$ and $L_2$).

*Proof.* We first note that if $L_1 \subseteq L_2$ are two extensions of $K$ then transitivity of the field norm (Corollary 4.48) implies

$$\mathrm{N}_{L_2/K} = \mathrm{N}_{L_1/K} \circ \mathrm{N}_{L_2/L_1},$$

and therefore $\mathrm{N}(L_2^\times) \subseteq \mathrm{N}(L_1^\times)$; thus the map $L \mapsto \mathrm{N}(L^\times)$ reverses inclusions.

This immediately implies $\mathrm{N}((L_1L_2)^\times) \subseteq \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times)$, since $L_1, L_2 \subseteq L_1L_2$. For the reverse inclusion, let us consider the commutative diagram

$$\begin{tikzcd} K^\times \arrow[r, "\theta_{L_1L_2/K}"] \arrow[dr, "\theta_{L_1/K}\times\theta_{L_2/K}"'] & \mathrm{Gal}(L_1L_2/K) \arrow[d, hook, "\mathrm{res}\times\mathrm{res}"] \\ & \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K) \end{tikzcd}$$

By Theorem 25.2, each $x \in \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) \subseteq K^\times$ lies in the kernel of $\theta_{L_1/K}$ and $\theta_{L_2/K}$, hence in the kernel of $\theta_{L_1L_2/K}$ (by the diagram), and therefore in $\mathrm{N}((L_1L_2)^\times)$ (by Theorem 25.2 again). This proves (a).

We now show that $L \mapsto \mathrm{N}(L^\times)$ is a bijection; it is surjective by definition, so we just need to show it is injective. If $\mathrm{N}(L_2^\times) = \mathrm{N}(L_1^\times)$ then by (a) we have

$$\mathrm{N}((L_1L_2)^\times) = \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) = \mathrm{N}(L_1^\times) = \mathrm{N}(L_2^\times),$$

and Theorem 25.2 implies $\mathrm{Gal}(L_1L_2/K) \simeq \mathrm{Gal}(L_1/K) \simeq \mathrm{Gal}(L_2/K)$, which forces $L_1 = L_2$; thus $L \mapsto \mathrm{N}(L^\times)$ is injective.

We now prove (b). The field $L_1 \cap L_2$ is the largest extension of $K$ that lies in both $L_1$ and $L_2$, while $\mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times)$ is the smallest subgroup of $K^\times$ containing both $\mathrm{N}(L_1^\times)$ and $\mathrm{N}(L_2^\times)$; they therefore correspond under the inclusion reversing bijection $L \mapsto \mathrm{N}(L^\times)$ and we have $\mathrm{N}((L_1 \cap L_2)^\times) = \mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times)$ as desired.

Finally, let us prove that every subgroup of $K^\times$ that contains a norm group is a norm group. Suppose $\mathrm{N}(L^\times) \subseteq H \subseteq K^\times$, for some finite abelian $L/K$, and subgroup $H$ of $K^\times$, and put $F := L^{\theta_{L/K}(H)}$. We have a commutative diagram

$$\begin{tikzcd} K^\times \arrow[r, "\theta_{L/K}"] \arrow[dr, "\theta_{F/K}"'] & \mathrm{Gal}(L/K) \arrow[d, "\mathrm{res}"] \\ & \mathrm{Gal}(F/K) \end{tikzcd}$$

in which $\mathrm{Gal}(L/F) = \theta_{L/K}(H)$ is precisely the kernel of the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(F/K)$ induced by restriction. It follows from Theorem 25.2 that

$$H = \theta_{L/K}^{-1}(\mathrm{Gal}(L/F)) = \mathrm{N}(F^\times)$$

is a norm group as claimed. $\square$

**Lemma 25.6.** *Let $L/K$ be an extension of local fields. If $\mathrm{N}(L^\times)$ has finite index in $K^\times$ then it is open.*

*Proof.* The lemma is clear if $K$ is archimedean, so let us assume it is nonarchimedean. Suppose $[K^\times : \mathrm{N}(L^\times)] < \infty$. The unit group $\mathcal{O}_L^\times$ is compact, so $\mathrm{N}(\mathcal{O}_L^\times)$ is compact (since $\mathrm{N}\colon L^\times \to K^\times$ is continuous) and therefore closed in the Hausdorff space $K^\times$. For any $\alpha \in L$,

$$\alpha \in \mathcal{O}_L^\times \iff |\alpha| = 1 \iff |\mathrm{N}_{L/K}(\alpha)| = 1 \iff \mathrm{N}_{L/K}(\alpha) \in \mathcal{O}_K^\times,$$

and therefore

$$\mathrm{N}(\mathcal{O}_L^\times) = \mathrm{N}(L^\times) \cap \mathcal{O}_K^\times.$$

It follows that $\mathrm{N}(\mathcal{O}_L^\times)$ is equal to the kernel of the map $\mathcal{O}_K^\times \hookrightarrow K^\times \twoheadrightarrow K^\times/\mathrm{N}(L^\times)$ and therefore $[\mathcal{O}_K^\times : \mathrm{N}(\mathcal{O}_L^\times)] \le [K^\times : \mathrm{N}(L^\times)] < \infty$. Thus $\mathrm{N}(\mathcal{O}_L^\times)$ is a closed subgroup of finite index in $\mathcal{O}_K^\times$, hence open (its complement is a finite union of closed cosets, hence closed), and $\mathcal{O}_K^\times$ is open in $K^\times$, so $\mathrm{N}(\mathcal{O}_L^\times)$ is open in $K^\times$.[1] $\square$

**Remark 25.7.** If $K$ is a local field of characteristic zero then one can show that in fact every finite index subgroup of $K^\times$ is open (whether it is a norm group or not), but this is not true in positive characteristic.

[1] Recall that in a nonarchimedean local field, $|K^\times|$ is discrete in $\mathbb{R}_{>0}$ and we can always pick $\epsilon > 0$ so that $\mathcal{O}_K^\times = \{x \in K^\times : 1 - \epsilon < |x| < 1 + \epsilon\}$, which is clearly open in the metric topology induced by $|\ |$.

## 25.3 The main theorems of local class field theory

It follows from local Artin reciprocity that all norm groups have finite index; Lemma 25.6 then implies that all norm groups are finite index open subgroups of $K^\times$. The existence theorem of local class field theory states that the converse also holds.

**Theorem 25.8** (Local Existence Theorem). *Let $K$ be a local field. For every finite index open subgroup $H$ of $K^\times$ there is a unique finite abelian extension $L/K$ inside $K^{\mathrm{ab}}$ for which $H = \mathrm{N}_{L/K}(L^\times)$.*

The local Artin homomorphism $\theta_K\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is not an isomorphism; indeed, it cannot be, because $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is compact but $K^\times$ is not. However, the local existence theorem implies that after taking profinite completions it becomes one. We can then summarize all of local class field theory in the following theorem.

**Theorem 25.9** (Main Theorem of Local Class Field Theory). *Let $K$ be a local field. The local Artin homomorphism induces a canonical isomorphism*

$$\widehat{\theta}_K\colon \widehat{K^\times} \xrightarrow{\sim} \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

*of profinite groups.*

*Proof.* The Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a profinite group, isomorphic to the inverse limit

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \varprojlim_L \mathrm{Gal}(L/K), \qquad (1)$$

where $L$ ranges over the finite extensions of $K$ in $K^{\mathrm{ab}}$ ordered by inclusion; see Theorem 24.21. It follows from Lemma 25.6, the local existence theorem (Theorem 25.8), and the definition of the profinite completion, that

$$\widehat{K^\times} \simeq \varprojlim_L K^\times/\mathrm{N}(L^\times), \qquad (2)$$

where $L$ ranges over finite abelian extensions of $K$ (in $K^{\mathrm{sep}}$). By local Artin reciprocity (Theorem 25.2), for each finite abelian extension $L/K$ we have an isomorphism

$$\theta_{L/K}\colon K^\times/\mathrm{N}(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K),$$

and these isomorphisms commute with the inclusion maps between finite abelian extensions of $K$. We thus have an isomorphism of the inverse systems appearing in (1) and (2). The isomorphism is canonical because the Artin map is unique and the isomorphisms in (1) and (2) are both canonical. $\square$

In view of Theorem 25.9, we would like to better understand the profinite group $\widehat{K^\times}$. If $K$ is archimedean then $\widehat{K^\times}$ is either trivial or the cyclic group of order 2, so let us assume that $K$ is nonarchimedean. If we pick a uniformizer $\pi$ for the maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, then we can uniquely write each $x \in K^\times$ in the form $u\pi^{v(x)}$, with $u \in \mathcal{O}_K^\times$ and $v(x) \in \mathbb{Z}$, and this defines an isomorphism

$$\begin{aligned} K^\times &\xrightarrow{\sim} \mathcal{O}_K^\times \times \mathbb{Z} \\ x &\longmapsto (x/\pi^{v(x)}, v(x)). \end{aligned}$$

Taking profinite completions (which commutes with products), we obtain an isomorphism

$$\widehat{K^\times} \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}},$$

since the unit group

$$\mathcal{O}_K^\times \simeq \mathbb{F}_\mathfrak{p}^\times \times \mathcal{O}_K \simeq \mathbb{F}_\mathfrak{p}^\times \times \varprojlim_n \mathcal{O}_K/\mathfrak{p}^n$$

is already profinite (hence isomorphic to its profinite completion, by Corollary 24.18). Note that the isomorphism $\widehat{K^\times} \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}$ is *not* canonical; it depends on our choice of $\pi$, and there are uncountably many $\pi$ to choose from.

We have a commutative diagram of exact sequences of topological groups

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d, "\wr"] & K^\times \arrow[r, "v"] \arrow[d, "\theta_K"] & \mathbb{Z} \arrow[r] \arrow[d, hook, "\phi"] & 0 \\
1 \arrow[r] & \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \arrow[r] & \mathrm{Gal}(K^{\mathrm{ab}}/K) \arrow[r, "\mathrm{res}"] & \mathrm{Gal}(K^{\mathrm{unr}}/K) \arrow[r] & 1
\end{tikzcd}$$

in which the bottom row is the profinite completion of the top row. The map $\phi$ on the right is given by

$$\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}} \simeq \mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p}) \simeq \mathrm{Gal}(K^{\mathrm{unr}}/K)$$

and sends 1 to the sequence of Frobenius elements $(\mathrm{Frob}_{L/K})$ in the profinite group

$$\mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \varprojlim_L \mathrm{Gal}(L/K) \subseteq \prod_L \mathrm{Gal}(L/K),$$

where $L$ ranges over finite unramified extensions of $K$; here we are using the canonical isomorphisms $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p})$ given by Theorem 10.16. The Frobenius element $\phi(1)$ is a *topological generator* for $\mathrm{Gal}(K^{\mathrm{unr}}/K)$, meaning that it generates a dense subset.

**Remark 25.10.** The Frobenius element $\phi(1) \in \mathrm{Gal}(K^{\mathrm{unr}}/K)$ corresponds to the Frobenius automorphism $x \mapsto x^{\#\mathbb{F}_\mathfrak{p}}$ of $\mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p})$; both are canonical topological generators of the Galois groups in which they reside, and both are sometimes referred to as the *arithmetic Frobenius*. There is another obvious generator for $\mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p})$, namely $\phi(-1)$, which is called the *geometric Frobenius* (for reasons we won't explain here).

The group $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \simeq \mathcal{O}_K^\times$ corresponds to the inertia subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}})$. The top sequence splits (but not canonically), hence so does the bottom, and we have

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \times \mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

For each choice of a uniformizer $\pi \in \mathcal{O}_K$ we get a decomposition $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$ corresponding to $K^\times = \mathcal{O}_K^\times \pi^{\mathbb{Z}}$. The field $K_\pi$ is the subfield of $K^{\mathrm{ab}}$ fixed by $\theta_K(\pi) \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Equivalently,

$$K_\pi = \bigcup_{\substack{\text{finite abelian} \\ \text{totally ramified} L/K \\ \text{for which } \pi \in \mathrm{N}(L^\times)}} L.$$

**Example 25.11.** Let $K = \mathbb{Q}_p$ and pick $\pi = p$. The decomposition $K = K_\pi K^{\mathrm{unr}}$ is

$$\mathbb{Q}_p^{\mathrm{ab}} = \bigcup_n \mathbb{Q}_p(\zeta_{p^n}) \cdot \bigcup_{m \perp p} \mathbb{Q}_p(\zeta_m),$$

where the first union on the RHS is fixed by $\theta_K(p)$ and the second is fixed by $\theta_K(\mathcal{O}_K^\times)$.

Constructing the local Artin homomorphism is the difficult part of local class field theory and we will not prove it in this course (but see 18.786). However, assuming the local existence theorem, it is easy to show that, if it exists, the local Artin homomorphism is unique.

**Proposition 25.12.** *Let $K$ be a local field and assume every finite index open subgroup of $K^\times$ is a norm group. There is at most one homomorphism $\theta\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ of topological groups that has the properties given in Theorem 25.2.*

*Proof.* Let $\mathfrak{p} = \langle \pi \rangle$ be the maximal ideal of $\mathcal{O}_K$, and for each integer $n \geq 0$ let $K_{\pi,n}/K$ be the finite abelian extension for which $\mathrm{N}(K_{\pi,n}^\times)$ is the norm group corresponding to the finite index subgroup $(1 + \mathfrak{p}^n)\langle \pi \rangle$ of $K^\times \simeq \mathcal{O}_K^\times \langle \pi \rangle$. Suppose $\theta\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a continuous homomorphism as in Theorem 25.2. Then $K_\pi = \bigcup_n K_{\pi,n}$, and $\theta(\pi)$ fixes $K_\pi$, since $\pi \in \mathrm{N}(K_{\pi,n}) = \ker \theta_{K_{\pi,n}/K}$ for all $n \geq 0$. We also know that $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for all finite unramified extensions $L/K$, which uniquely determines the action of $\theta(\pi)$ on $K^{\mathrm{unr}}$, and hence on $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$.

Now suppose $\theta'\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is another continuous homomorphism satisfying the properties in Theorem 25.2. By the argument above we must have $\theta'(\pi) = \theta(\pi)$ for every uniformizer $\pi$ of $\mathcal{O}_K$, and $K^\times$ is generated by its subset of uniformizers: if we fix one uniformizer $\pi$, every $x \in K^\times$ can be written as $u\pi^n = (u\pi)\pi^{n-1}$ for some $u \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$, and $u\pi$ is another uniformizer). It follows that $\theta(x) = \theta'(x)$ for all $x \in K^\times$ and therefore $\theta = \theta'$ is unique. □

**Remark 25.13.** One approach to proving local class field theory uses the theory of formal groups due to Lubin and Tate to explicitly construct the fields $K_\pi = \bigcup_n K_{\pi,n}$ in the proof of Proposition 25.12, along with a continuous homomorphism $\theta_\pi\colon \mathcal{O}_K^\times \to \mathrm{Gal}(K_\pi/K)$ that extends uniquely to a continuous homomorphism $\theta\colon K^\times \to \mathrm{Gal}(K_\pi K^{\mathrm{unr}}/K)$. One then shows that $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$ (using the Hasse-Arf Theorem), and that $\theta$ does not depend on the choice of $\pi$; see [1, §I.2-4] for details.

### 25.4 Finite abelian extensions

Local class field theory gives us canonical bijections between the following sets:

(1) finite-index open subgroups of $K^\times$ (all of which are necessarily normal);

(2) open subgroups of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ (which are necessarily normal and of finite index);

(3) finite abelian extensions of $K$ in $K^{\mathrm{ab}}$ (which necessarily lie in $K^{\mathrm{ab}}$).

The bijection from (1) to (2) is induced by the isomorphism $\widehat{K^\times} \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K)$ given by Theorem 25.9 and is inclusion preserving. The bijection from (2) to (3) follows from Galois theory (for infinite extensions), and is inclusion reversing, while the bijection from (3) to (1) is via the map $L \mapsto \mathrm{N}(L^\times)$, which is also inclusion reversing.

## References


[1] J.S. Milne, *Class field theory*, version 4.02, 2013.

[2] Jean-Pierre Serre, *Local fields*, Springer, 1979.