23 The ring of adeles, strong approximation

23.1 Introduction to adelic rings

Recall that we have a canonical injection

$$\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z} \simeq \prod_p \mathbb{Z}_p,$$

that embeds $\mathbb{Z}$ into the product of its nonarchimedean completions. Each of the rings $\mathbb{Z}_p$ is compact, hence $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is compact (by Tychonoff's theorem). If we consider the analogous product $\prod_p \mathbb{Q}_p$ of the completions of $\mathbb{Q}$, each of the local fields $\mathbb{Q}_p$ is locally compact (as is the archimedean field $\mathbb{Q}_\infty = \mathbb{R}$), but the product $\prod_p \mathbb{Q}_p$ is **not locally compact**. Local compactness is important to us, because it gives us a Haar measure (recall that every locally compact group has a translation-invariant measure that is unique up to scaling), a tool we would like to have at our disposal.

To see where the problem arises, recall that for any family of topological spaces $(X_i)_{i\in I}$ (here the index set I may be any set), the product topology on $X := \prod X_i$ is, by definition, the weakest topology that makes all the projection maps $\pi_i \colon X \to X_i$ continuous; this implies that it is generated by open sets of the form $\pi_i^{-1}(U_i)$ with $U_i \subseteq X_i$ open. Thus every open set in X is a (possibly empty or infinite) union of open sets of the form

$$\prod_{i\in S} U_i \times \prod_{i\notin S} X_i,$$

with $S \subseteq I$ finite and each $U_i \subseteq X_i$ open (these sets form a *basis* for the topology on X). In particular, every open set $U \subseteq X$ satisfies $\pi_i(U) = X_i$ for all but finitely many $i \in I$. Unless all but finitely many of the X_i are compact, the space X cannot possibly be locally compact for the simple reason that no compact set C in X contains a nonempty open set (if it did then we would have $\pi_i(C) = X_i$ compact for all but finitely many $i \in I$). Recall that for X to be locally compact means that every $x \in X$ we have $x \in U \subseteq C$ for some open set U and compact set C (so C is a compact neighborhood of x).

To solve this problem we want to take the product of the fields $\mathbb{Q}_p$ (or more generally, the completions of any global field) in a different way, one that yields a locally compact topological ring. This is the motivation of the *restricted product*, a topological construction that was invented primarily for the purpose of solving this number-theoretic problem.

23.2 Restricted products

This section is purely about the topology of restricted products; readers familiar with restricted products should feel free to skip to the next section.

Definition 23.1. Let (X_i) be a family of topological spaces indexed by $i \in I$, and let (U_i) be a family of open sets $U_i \subseteq X_i$. The *restricted product* $\prod(X_i, U_i)$ is the topological space

$$\prod(X_i, U_i) := \{(x_i) : x_i \in U_i \text{ for almost all } i \in I\} \subseteq \prod X_i$$

with the basis of open sets

$$\mathcal{B} := \left\{ \prod V_i : V_i \subseteq X_i \text{ is open for all } i \in I \text{ and } V_i = U_i \text{ for almost all } i \in I \right\},$$

where *almost all* means all but finitely many.

For each $i \in I$ we have a projection map $\pi_i \colon \prod(X_i, U_i) \to X_i$ defined by $(x_i) \mapsto x_i$; each π_i is continuous, since if U_i is an open subset of X_i, then $\pi_i^{-1}(U_i)$ is the union of all $V = \prod V_i \in \mathcal{B}$ with $V_i = U_i$, which is an open set.

As sets, we always have

$$\prod U_i \subseteq \prod(X_i, U_i) \subseteq \prod X_i,$$

but in general the restricted product topology on $\prod(X_i, U_i)$ is not the same as the subspace topology it inherits from $\prod X_i$; it has more open sets. For example, $\prod U_i$ is an open set in $\prod(X_i, U_i)$, but unless $U_i = X_i$ for almost all i, in which case $\prod(X_i, U_i) = \prod X_i$ (both as sets and as topological spaces), it is not open in $\prod X_i$, and it is not open in the subspace topology on $\prod(X_i, U_i)$ because it does not contain the intersection of $\prod(X_i, U_i)$ with any basic open set in $\prod X_i$ (recall that these look like $\prod V_i$ with $V_i = X_i$ for almost all i), so it cannot be equal to the intersection of $\prod(X_i, U_i)$ with any open set in $\prod X_i$.

Thus the restricted product is a generalization of the direct product, and the two coincide if and only if $U_i = X_i$ for almost all i. This is automatically true when the index set I is finite, so only infinite restricted products are interesting.

Remark 23.2. The restricted product does not depend on any particular U_i. Indeed,

$$\prod(X_i, U_i) = \prod(X_i, U_i')$$

whenever $U_i' = U_i$ for almost all i; note that the two restricted products are not merely isomorphic, they are identical, both as sets and as topological spaces. It is thus enough to specify the U_i for all but finitely many $i \in I$.

Each $x \in X := \prod(X_i, U_i)$ determines a finite subset $S(x) \subseteq I$ consisting of the indices i for which $x_i \notin U_i$ (which may be the empty set). Given any finite $S \subseteq I$ we may consider

$$X_S := \{x \in X : S(x) = S\} = \prod_{i \in S} X_i \times \prod_{i \notin S} U_i.$$

Notice that $X_S \in \mathcal{B}$ is an open set, and we can view it as a topological space in two ways, both as a subspace of X or as a direct product of certain X_i and U_i. But notice that restricting the basis $\mathcal{B}$ for X to a basis for the subspace X_S yields

$$\mathcal{B}_S := \left\{ \prod V_i : V_i \subseteq \pi_i(X_S) \text{ is open and } V_i = U_i = \pi_i(X_S) \text{ for almost all } i \in I \right\},$$

which is the standard basis for the product topology, so the two topologies on X_S coincide.

We have $X_S \subseteq X_T$ whenever $S \subseteq T$, thus if we partially order the finite subsets $S \subseteq I$ by inclusion, the family of topological spaces $\{X_S : S \subseteq I \text{ finite}\}$ with inclusion maps $\{i_{ST} \colon X_S \hookrightarrow X_T | S \subseteq T\}$ forms a *direct system*, and we have a corresponding *direct limit*

$$\varinjlim_S X_S := \coprod X_S / \sim,$$

which is the quotient of the coproduct space (disjoint union) $\coprod X_S$ by the equivalence relation $x \sim i_{ST}(x)$ for all $x \in S \subseteq T$.[1] This direct limit is canonically isomorphic to the restricted product X, which gives us another way to define the restricted product; before proving this let us recall the general definition of a direct limit of topological spaces.

[1]The topology on $\coprod X_S$ is the weakest topology that makes the injections $X_S \hookrightarrow \coprod X_S$ continuous; its open sets are disjoint unions of open sets in the X_S. The topology on $\coprod X_S / \sim$ is the weakest topology that makes the quotient map $\coprod X_S \to \coprod X_S / \sim$ continuous; its open sets are images of open sets in $\coprod X_S$.

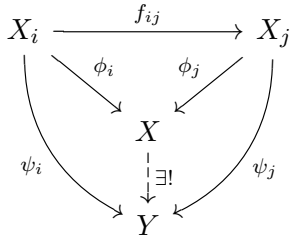
Definition 23.3. A *direct system* (or *inductive system*) in a category is a family of objects $\{X_i : i \in I\}$ indexed by a directed set I (see Definition 8.7) and a family of morphisms $\{f_{ij} \colon X_i \to X_j : i \le j\}$ such that each f_{ii} is the identity and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \le j \le k$.

Definition 23.4. Let (X_i, f_{ij}) be a direct system of topological spaces. The *direct limit* (or *inductive limit*) of (X_i, f_{ij}) is the quotient space

$$X = \varinjlim X_i := \coprod_{i \in I} X_i / \sim,$$

where $x_i \sim f_{ij}(x_i)$ for all $i \le j$. It is equipped with continuous maps $\phi_i \colon X_i \to X$ that are compositions of the inclusion maps $X_i \hookrightarrow \coprod X_i$ and quotient maps $\coprod X_i \twoheadrightarrow \coprod X_i / \sim$ and satisfy $\phi_i = \phi_j \circ f_{ij}$ for $i \le j$.

The topological space $X = \varinjlim X_i$ has the universal property that if Y is another topological space with continuous maps $\psi_i \colon X_i \to Y$ that satisfy $\psi_i = \psi_j \circ f_{ij}$ for $i \le j$, then there is a unique continuous map $X \to Y$ for which all of the diagrams



commute (this universal property defines the direct limit in any category with coproducts).

We now prove that that $\prod(X_i, U_i) \simeq \varinjlim X_S$ as claimed above.

Proposition 23.5. *Let (X_i) be a family of topological spaces indexed by $i \in I$, let (U_i) be a family of open sets $U_i \subseteq X_i$, and let $X := \prod(X_i, U_i)$ be the corresponding restricted product. For each finite $S \subseteq I$ define*

$$X_S := \prod_{i \in S} X_i \times \prod_{i \notin S} U_i \subseteq X,$$

and inclusion maps $i_{ST} \colon X_S \hookrightarrow X_T$, and let $\varinjlim X_S$ be the corresponding direct limit.

There is a canonical homeomorphism of topological spaces

$$\varphi \colon X \xrightarrow{\sim} \varinjlim X_S$$

that sends $x \in X$ to the equivalence class of $x \in X_{S(x)} \subseteq \coprod X_S$ in $\varinjlim X_S := \coprod X_S / \sim$, where $S(x) := \{i \in I : x_i \notin U_i\}$.

Proof. To prove that the map $\varphi \colon X \to \varinjlim X_S$ is a homeomorphism, we need to show that it is (1) a bijection, (2) continuous, and (3) an open map.

(1) For each equivalence class $\mathcal{C} \in \varinjlim X_S := \coprod X_S / \sim$, let $S(\mathcal{C})$ be the intersection of all the sets S for which $\mathcal{C}$ contains an element of $\coprod X_S$ in X_S. Then $S(x) = S(\mathcal{C})$ for all $x \in \mathcal{C}$, and $\mathcal{C}$ contains a unique element for which $x \in X_{S(x)} \subseteq \coprod X_S$ (distinct $x, y \in X_S$ cannot be equivalent). Thus φ is a bijection.

(2) Let U be an open set in $\varinjlim X_S = \coprod X_S / \sim$. The inverse image V of U in $\coprod X_S$ is open, as are the inverse images V_S of V under the canonical injections $\iota \colon X_S \hookrightarrow \coprod X_S$. The union of the V_S in X is equal to $\varphi^{-1}(U)$ and is an open set in X; thus φ is continuous.

(3) Let U be an open set in X. Since the X_S form an open cover of X, we can cover U with open sets $U_S := U \cap X_S$, and then $\coprod U_S$ is an open set in $\coprod X_S$. Moreover, for each $x \in \coprod U_S$, if $y \sim x$ for some $y \in \coprod X_S$ then y and x must correspond to the same element in U; in particular, $y \in \coprod U_S$, so $\coprod U_S$ is a union of equivalence classes in $\coprod X_S$. It follows that its image in $\varinjlim X_S = \coprod X_S / \sim$ is open. □

Proposition 23.5 gives us another way to construct the restricted product $\prod(X_i, U_i)$: rather than defining it as a subset of $\prod X_i$ with a modified topology, we can instead construct it as a limit of direct products that are subspaces of $\prod X_i$.

We now specialize to the case of interest, where we are forming a restricted product using a family $(X_i)_{i\in I}$ of locally compact spaces and a family of open subsets (U_i) that are almost all compact. Under these conditions the restricted product $\prod(X_i, U_i)$ is locally compact, even though the product $\prod X_i$ is not unless the index set I is finite.

Proposition 23.6. *Let $(X_i)_{i\in I}$ be a family of locally compact topological spaces and let $(U_i)_{i\in I}$ be a corresponding family of open subsets $U_i \subseteq X_i$ almost all of which are compact. Then the restricted product $X := \prod(X_i, U_i)$ is locally compact.*

Proof. We first note that for each finite set $S \subseteq I$ the topological space

$$X_S := \prod_{i\in S} X_i \times \prod_{i\notin S} U_i$$

can be viewed as a finite product of locally compact spaces, since all but finitely many of the U_i are compact and the product of these is compact (by Tychonoff's theorem), hence locally compact. A finite product of locally compact spaces is always locally compact, since we can construct compact neighborhoods as products of compact neighborhoods in each factor (the key point is that in a finite product, products of open sets are open); thus the X_S are all locally compact, and the X_S cover X (since each $x \in X$ lies in $X_{S(x)}$). It follows that X is locally compact, since each $x \in X_S$ has a compact neighborhood $x \in U \subseteq C \subseteq X_S$ that is also a compact neighborhood in X (the image of C under the inclusion map $X_S \to X$ is certainly compact, and U is open in X because X_S is open in X). □

23.3 The ring of adeles

Recall that for a global field K (finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$), we use M_K to denote the set of places of K (equivalence classes of absolute values), and for any $v \in M_K$ we use K_v to denote the corresponding local field (the completion of K with respect to v). When v is nonarchimedean we use $\mathcal{O}_v$ to denote the valuation ring of K_v, and for nonarchimedean v we define $\mathcal{O}_v := K_v$.[2]

Definition 23.7. Let K be a global field. The *adele ring*[3] of K is the restricted product

$$\mathbb{A}_K := \prod(K_v, \mathcal{O}_v)_{v\in M_K},$$

which we may view as a subset (but not a subspace!) of $\prod_v K_v$; indeed

$$\mathbb{A}_K = \left\{(a_v) \in \prod K_v : a_v \in \mathcal{O}_v \text{ for almost all } v\right\}.$$

[2]Per Remark 23.2, as far as the topology goes it doesn't matter how we define $\mathcal{O}_v$ at the finite number of archimedean places, but we would like each $\mathcal{O}_v$ to be a topological ring, which motivates this choice.

[3]In French one writes *adèle*, but it is common practice to omit the accent when writing in English.

For each $a \in \mathbb{A}_K$ we use a_v to denote its projection in K_v; we make $\mathbb{A}_K$ a ring by defining addition and multiplication component-wise.

For each finite set of places S we have the subring of *S-adeles*

$$\mathbb{A}_{K,S} := \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v,$$

which is a direct product of topological rings. By Proposition 23.5, $\mathbb{A}_K \simeq \varinjlim \mathbb{A}_{K,S}$ is the direct limit of the S-adele rings, which makes it clear that $\mathbb{A}_K$ is also a topological ring.

The canonical embeddings $K \hookrightarrow K_v$ induce a canonical embedding

$$\begin{aligned} K &\hookrightarrow \mathbb{A}_K \\ x &\mapsto (x, x, x, \ldots). \end{aligned}$$

Note that for each $x \in K$ we have $x \in \mathcal{O}_v$ for all but finitely many v. The image of K in $\mathbb{A}_K$ forms the subring of *principal adeles* (which of course is also a field).

We extend the normalized absolute value $\| \ \|_v$ of K_v (see Definition 13.18) to $\mathbb{A}_K$ via

$$\|a\|_v := \|a_v\|_v,$$

and define the *adelic absolute value* (or *adelic norm*)

$$\|a\| := \prod_{v \in M_K} \|a\|_v \in \mathbb{R}_{\geq 0}$$

which we note converges because $\|a\|_v \leq 1$ for almost all v. For $\|a\| \neq 0$ this is equal to the size of the *M_K-divisor* ($\|a\|_v$) we defined in Lecture 15 (see Definition 15.1). For any nonzero principal adele a we necessarily have $\|a\| = 1$, by the product formula (Theorem 13.22).

Example 23.8. For $K = \mathbb{Q}$ the adele ring $\mathbb{A}_\mathbb{Q}$ is the union of the rings

$$\mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$$

where S varies over finite sets of primes (but note that the topology is the restricted product topology, not the subspace topology in $\prod_{p \leq \infty} \mathbb{Q}_p$). We can also write $\mathbb{A}_\mathbb{Q}$ as

$$\mathbb{A}_\mathbb{Q} = \left\{ a \in \prod_{p \leq \infty} \mathbb{Q}_p : \|a\|_p \leq 1 \text{ for almost all } p \right\}.$$

Proposition 23.9. *The adele ring $\mathbb{A}_K$ of a global field K is locally compact and Hausdorff.*

Proof. Local compactness follows from Proposition 23.6, since the local fields K_v are all locally compact and all but finitely many $\mathcal{O}_v$ are valuation rings of a nonarchimedean local field, hence compact ($\mathcal{O}_v = \{x \in K_v : \|x\|_v \leq 1\}$ is a closed ball in a locally compact metric space). If $x, y \in \mathbb{A}_K$ are distinct then $x_v \neq y_v$ for some $v \in M_K$, and since K_v is Hausdorff we can separate x_v and y_v by open sets whose inverse images under the projection map $\pi_v \colon \mathbb{A}_K \to K_v$ are open sets separating x and y; thus $\mathbb{A}_K$ is Hausdorff. □

Proposition 23.9 implies that the additive group of $\mathbb{A}_K$ (which is sometimes denoted $\mathbb{A}_K^+$ to emphasize that we are viewing it as a group rather than a ring) is a locally compact group, and therefore has a Haar measure that is unique up to scaling. Each of the completions K_v is a local field with a Haar measure μ_v that we normalize as follows:

- $\mu_v(\mathcal{O}_v) = 1$ for all nonarchimedean v;
- $\mu_v(S) = \mu_{\mathbb{R}}(S)$ for $K_v \simeq \mathbb{R}$, where $\mu_{\mathbb{R}}(S)$ is the Lebesgue measure on $\mathbb{R}$;
- $\mu_v(S) = 2\mu_{\mathbb{C}}(S)$ for $K_v \simeq \mathbb{C}$, where $\mu_{\mathbb{C}}(S)$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$.

Note that the normalization of μ_v at the archimedean places is consistent with the measure μ on $K_{\mathbb{R}} \simeq \mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$ induced by the canonical inner product on $K_{\mathbb{R}} \subseteq K_{\mathbb{C}}$ that we defined in Lecture 14 (see §14.2).

We now define a measure μ on $\mathbb{A}_K$ as follows. We take as a basis for the σ-algebra of measurable sets all sets of the form $\prod_v B_v$, where each B_v is a measurable set in K_v with $\mu_v(B_v) < \infty$ such that $B_v = \mathcal{O}_v$ for almost all v (the σ-algebra is then generated by taking countable intersections, unions, and complements in $\mathbb{A}_K$). We then define

$$\mu\left(\prod_v B_v\right) := \prod_v \mu_v(B_v).$$

It is easy to verify that μ is a Radon measure, and it is clearly translation invariant since each of the Haar measures μ_v is translation invariant and addition is defined componentwise; note that for any $x \in \mathbb{A}_K$ and measurable set $B = \prod_v B_v$ the set $x + B = \prod_v (x_v + B_v)$ is also measurable, since $x_v + B_v = \mathcal{O}_v$ whenever $x_v \in \mathcal{O}_v$ and $B_v = \mathcal{O}_v$, and this applies to almost all v. It follows from uniqueness of the Haar measure (up to scaling) that μ is a Haar measure on $\mathbb{A}_K$ which we henceforth adopt as our normalized Haar measure on $\mathbb{A}_K$.

We now want to understand the behavior of the adele ring $\mathbb{A}_K$ under base change. Note that the canonical embedding $K \hookrightarrow \mathbb{A}_K$ makes $\mathbb{A}_K$ a K-vector space, and if L/K is any finite separable extension of K (also a K-vector space), we may consider the tensor product

$$\mathbb{A}_K \otimes L,$$

which is also an L-vector space. As a topological K-vector space, the topology on $\mathbb{A}_K \otimes L$ is just the product topology on $[L : K]$ copies of of $\mathbb{A}_K$ (this applies whenever we take a tensor product of topological vector spaces, one of which has finite dimension).

Proposition 23.10. *Let L be a finite separable extension of a global field K. There is a natural isomorphism of topological rings*

$$\mathbb{A}_L \simeq \mathbb{A}_K \otimes_K L$$

that makes the following diagram commute

$$\begin{array}{ccc} L & \xrightarrow{\sim} & K \otimes_K L \\ \downarrow & & \downarrow \\ \mathbb{A}_L & \xrightarrow{\sim} & \mathbb{A}_K \otimes_K L \end{array}$$

Proof. On the RHS the tensor product $\mathbb{A}_K \otimes_K L$ is isomorphic to the restricted product

$$\prod\nolimits_v (K_v \otimes_K L, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L).$$

Explicitly, each element of $\mathbb{A}_K \otimes_K L$ is a finite sum of elements of the form $(a_v) \otimes x$, where $(a_v) \in \mathbb{A}_K$ and $x \in L$, and there is a natural isomorphism

$$\begin{aligned} \mathbb{A}_K \otimes_K L &\xrightarrow{\sim} \prod\nolimits_{v \in M_K} (K_v \otimes_K L, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L) \\ (a_v) \otimes x &\mapsto (a_v \otimes x) \end{aligned}$$

that is both a ring isomorphism and a homeomorphism of topological spaces.

On the LHS we have $\mathbb{A}_L := \prod_{w \in M_L} (L_w, \mathcal{O}_w)$. But note that $K_v \otimes_K L \simeq \prod_{w|v} L_w$, by Theorem 11.20 and $\mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L \simeq \prod_{w|v} \mathcal{O}_w$, by Corollary 11.23. These isomorphisms preserve both the algebraic and the topological structures of both sides, and it follows that

$$\mathbb{A}_K \otimes_K L \simeq \prod_{v \in M_K} (K_v \otimes_K L, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L) \simeq \prod_{w \in M_L} (L_w, \mathcal{O}_w) = \mathbb{A}_L$$

is an isomorphism of topological rings. The image of $x \in L$ in $\mathbb{A}_K \otimes_K L$ via the canonical embedding of L into $\mathbb{A}_K \otimes_K L$ is $1 \otimes x = (1, 1, 1, \ldots) \otimes x$, whose image $(x, x, x, \ldots) \in \mathbb{A}_L$ is equal to the image of $x \in L$ under the canonical embedding of L into its adele ring $\mathbb{A}_L$. □

Corollary 23.11. *Let L be a finite separable extension of a global field K of degree n. There is a natural isomorphism of topological K-vector spaces (and locally compact groups)*

$$\mathbb{A}_L \simeq \mathbb{A}_K \oplus \cdots \oplus \mathbb{A}_K$$

that identifies $\mathbb{A}_K$ with the direct sum of n copies of $\mathbb{A}_K$, and this isomorphism restricts to an isomorphism $L \simeq K \oplus \cdots \oplus K$ of the principal adeles of $\mathbb{A}_L$ with the n-fold direct sum of the principal adeles of $\mathbb{A}_K$.

Theorem 23.12. *For each global field L the principal adeles $L \subseteq \mathbb{A}_L$ form a discrete cocompact subgroup of the additive group of the adele ring $\mathbb{A}_L$.*

Proof. Let K be the rational subfield of L (so $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$). It follows from the previous corollary, that if the theorem holds for K then it holds for L, so we will prove the theorem for K. Let us identify K with its image in $\mathbb{A}_K$ (the principal adeles).

To show that the topological group K is discrete in $\mathbb{A}_K$, it suffices to show that 0 is an isolated point. Consider the open set

$$U = \{a \in \mathbb{A}_K : \|a\|_\infty < 1 \text{ and } \|a\|_v \le 1 \text{ for all } v < \infty\},$$

where ∞ denotes the unique infinite place of K (either the real place of $\mathbb{Q}$ or the place corresponding to the degree valuation $v_\infty(f/g) = \deg f - \deg g$ of $\mathbb{F}_q(t)$). The product formula (Theorem 13.22) implies $\|a\| = 1$ for all nonzero $a \in K \subseteq \mathbb{A}_K$, so $U \cap K = \{0\}$.

To prove that the quotient $\mathbb{A}_K/K$ is compact, we consider the set

$$W := \{a \in \mathbb{A}_K : \|a\|_v \le 1 \text{ for all } v\}.$$

If we let $U_\infty := \{x \in K_\infty : \|x\|_\infty \le 1\}$, then

$$W = U_\infty \times \prod_{v < \infty} \mathcal{O}_v \subseteq \mathbb{A}_{K,\{\infty\}} \subseteq \mathbb{A}_K$$

is a product of compact sets and therefore compact. We will show that W contains a complete set of coset representatives for K in $\mathbb{A}_K$. This implies that $\mathbb{A}_K/K$ is the image of the compact set W under the (continuous) quotient map $\mathbb{A}_K \to \mathbb{A}_K/K$, hence compact.

Let $a = (a_v)$ be any element of $\mathbb{A}_K$. We wish to show that $a = b + c$ for some $b \in W$ and $c \in K$, which we will do by constructing $c \in K$ so that $b = a - c \in W$.

For each $v < \infty$ define $x_v \in K$ as follows: put $x_v := 0$ if $\|a_v\| \le 1$ (almost all v), and otherwise choose $x_v \in K$ so that $\|a_v - x_v\|_v \le 1$ and $\|x_v\|_w \le 1$ for $w \ne v$. To show that such an x_v exists, let us first suppose $a_v = r/s \in K$ with $r, s \in \mathcal{O}_K$ coprime, and let $\mathfrak{p}$ be the

maximal ideal of $\mathcal{O}_v$. The ideals $\mathfrak{p}^{v(s)}$ and $\mathfrak{p}^{-v(s)}(s)$ are coprime, so we can write $r = r_1 + r_2$ with $r_1 \in \mathfrak{p}^{v(s)}$ and $r_2 \in \mathfrak{p}^{-v(s)}(s) \subseteq \mathcal{O}_K$, so that $a_v = r_1/s + r_2/s$ with $v(r_1/s) \geq 0$ and $w(r_2/s) \geq 0$ for all $w \neq v$. If we now put $x_v := r_2/s$, then $\|a_v - x_v\|_v = \|r_1/s\|_v \leq 1$ and $\|x_v\|_w = \|r_2/s\|_w \leq 1$ for all $w \neq v$ as desired. We now note that we can approximate $a_v \in K_v$ by $a'_v \in K$ with $\|a_v - a'_v\| < \epsilon$ and construct x_v using a'_v as above so that $\|a'_v - x_v\| \leq 1$ and $\|a_v - x_v\| \leq 1 + \epsilon$, but for sufficiently small ϵ this implies $\|a_v - x_v\| \leq 1$ (because $\| \ \|_v$ is discrete).

If we now consider $a_v \in K_v$ and a sequence of elements $a_i \in K$ that converge to a_v in K_v, the corresponding sequence of $x_i \in K$ constructed as above lies in a compact set and therefore contains a convergent subsequence whose limit we can take as our choice of x_v.

Finally, let $x := \sum_{v<\infty} x_v \in K$ and choose $x_\infty \in \mathcal{O}_K$ so that

$$\|a_\infty - x - x_\infty\|_\infty \leq 1.$$

For $K = \mathbb{Q}$ we can take $x_\infty \in \mathbb{Z}$ to be the nearest integer to the rational number $a_\infty - x$, and when $K = \mathbb{F}_q(t)$, if $a_\infty - x = f/g$ with $f, g \in \mathbb{F}_q[t]$ coprime, we can write $f = gh + u$ for some $h, u \in \mathbb{F}_q[t]$ with $\deg u < \deg g$ and then take $x_\infty := -h$.

Now let $c := \sum_{v\leq\infty} x_v \in K \subseteq \mathbb{A}_K$, and let $b := a - c$. Then $a = b + c$, with $c \in K$, and we claim that $b \in W$. For each $v < \infty$ we have $x_w \in \mathcal{O}_v$ for all $w \neq v$ and

$$\|b\|_v = \|a - c\|_v = \left\| a_v - \sum_{w\leq\infty} x_w \right\|_v \leq \max\left(\|a_v - x_v\|_v, \max(\{\|x_w\|_v : w \neq v\})\right) \leq 1,$$

by the nonarchimedean triangle inequality. For $v = \infty$ we have $\|b\|_\infty = \|a_\infty - c\|_\infty \leq 1$ by our choice of x_∞, and $\|b\|_v \leq 1$ for all v, so $b \in W$ as claimed and the theorem follows. □

23.4 Strong approximation

We are now ready to prove the strong approximation theorem, an important result that has many applications. In order to prove it we first prove an adelic version of the Blichfeldt-Minkowski lemma.

Lemma 23.13 (Blichfeldt-Minkowski lemma)**.** *Let K be a global field. There is a positive constant B such that for any $a \in \mathbb{A}_K$ with $\|a\| > B$ there exists a nonzero principal adele $x \in K \subseteq \mathbb{A}_K$ for which $\|x\|_v \leq \|a\|_v$ for all $v \in M_K$.*

Proof. Let $b_0 := \mathrm{covol}(K)$ be the measure of a fundamental region for K in $\mathbb{A}_K$ under our normalized Haar measure μ on $\mathbb{A}_K$ (by Theorem 23.12, K is cocompact, so b_0 is finite). Now define

$$b_1 := \mu\left(\left\{z \in \mathbb{A}_K : \|z\|_v \leq 1 \text{for all } v \text{ and } \|z\|_v \leq \tfrac{1}{4} \text{ if } v \text{ is archimedean}\right\}\right).$$

Then $b_1 \neq 0$, since K has only finitely many archimedean places, and we put $B := b_0/b_1$.[4]

Suppose $a \in \mathbb{A}_K$ satisfies $\|a\| > B$. We know that $\|a\|_v \leq 1$ for all almost all v, so $\|a\| > B$ implies that $\|a\|_v = 1$ for almost all v. Let us now consider the set

$$T := \left\{t \in \mathbb{A}_K : \|t\|_v \leq \|a\|_v \text{ for all } v \text{ and } \|t\|_v \leq \tfrac{1}{4}\|a\|_v \text{ if } v \text{ is archimedean}\right\}.$$

[4]With our canonical normalization of μ we will actually get the same B for all K, but we don't need this. With a little more care one can show that in fact $B = 1$ works.

From the definition of b_1 we have

$$\mu(T) = b_1\|a\| > b_1 B = b_0;$$

this follows from the fact that the Haar measure on $\mathbb{A}_K$ is the product of the normalized Haar measures μ_v on each of the K_v. Since $\mu(T) > b_0$, the set T is not contained in any fundamental region for K, so there must be distinct $t_1, t_2 \in T$ with the same image in $\mathbb{A}_K/K$, equivalently, whose difference $x = t_1 - t_2$ is a nonzero element of $K \subseteq \mathbb{A}_K$. Now

$$\|t_1 - t_2\|_v \le \begin{cases} \max(\|t_1\|_v, \|t_2\|_v) \le \|a\|_v & \text{nonarch. } v; \\ \|t_1\|_v + \|t_2\|_v \le 2 \cdot \frac{1}{4}\|x\|_v \le \frac{1}{2}\|a\|_v & \text{real } v; \\ (\|t_1 - t_2\|_v^{1/2})^2 \le (\|t_1\|_v^{1/2} + \|t_2\|_v^{1/2})^2 \le (2 \cdot \frac{1}{2}\|a\|_v^{1/2})^2 \le \|a\|_v & \text{complex } v. \end{cases}$$

Here we have used the fact that the normalized absolute value $\| \ \|_v$ satisfies the nonarchimedean triangle inequality when v is nonarchimedean, $\| \ \|_v$ satisfies the archimedean triangle inequality when v is real, and $\| \ \|_v^{1/2}$ satisfies the archimedean triangle inequality when v is complex. Thus $\|x\|_v = \|t_1 - t_2\|_v \le \|a\|_v$ for all places $v \in M_K$ as desired. □

Theorem 23.14 (Strong Approximation)**.** *Let $M_K = S \sqcup T \sqcup \{w\}$ be a partition of the places of a global field K with S finite. Given any $a_v \in K$ and $\epsilon_v \in \mathbb{R}_{>0}$ with $v \in S$, there exists an $x \in K$ for which*

$$\|x - a_v\|_v \le \epsilon_v \text{ for all } v \in S,$$
$$\|x\|_v \le 1 \text{ for all } v \in T,$$

(note that there is no constraint on $\|x\|_w$).

Proof. Let $W = \{z \in \mathbb{A}_K : \|z\|_v \le 1 \text{ for all } v \in M_K\}$ as in the proof of Theorem 23.12. Then W contains a complete set of coset representatives for $K \subseteq \mathbb{A}_K$, so $\mathbb{A}_K = K + W$. For any nonzero $u \in K \subseteq \mathbb{A}_K$ we also have $\mathbb{A}_K = K + uW$: given $c \in \mathbb{A}_K$ write $u^{-1}c \in \mathbb{A}_K$ as $u^{-1}c = a + b$ with $a \in K$ and $b \in W$ and then $c = ua + ub$ with $ua \in K$ and $ub \in uW$. Now choose $z \in \mathbb{A}_K$ such that

$$0 < \|z\|_v \le \epsilon_v \text{ for } v \in S, \quad 0 < \|z\|_v \le 1 \text{ for } v \in T, \quad \|z\|_w > B \prod_{v \ne w} \|z\|_v^{-1},$$

where B is the constant in the Blichfeldt-Minkowski Lemma 23.13 (this is clearly possible). We have $\|z\| > B$, so there is a nonzero $u \in K \subseteq \mathbb{A}_K$ with $\|u\|_v \le \|z\|_v$ for all $v \in M_K$.

Now let $a = (a_v) \in \mathbb{A}_K$ be the adele with a_v given by the hypothesis of the theorem for $v \in S$ and $a_v = 0$ for $v \notin S$. We have $\mathbb{A}_K = K + uW$, so $a = x + y$ for some $x \in K$ and $y \in uW$. Therefore

$$\|x - a_v\|_v = \|y\|_v \le \|u\|_v \le \|z\|_v \le \begin{cases} \epsilon_v & \text{for } v \in S, \\ 1 & \text{for } v \in T, \end{cases}$$

as desired. □

Corollary 23.15. *Let K be a global field and let w be any place of K. Then K is dense in the restricted product $\prod_{v \ne w}(K_v, \mathcal{O}_v)$.*

Remark 23.16. Theorem 23.14 and its corollary can be generalized to algebraic groups (the global field K can be viewed as the algebraic group $\mathrm{GL}_1(K)$); see [1] for a survey.

References

[1] Andrei S. Rapinchuk, *Strong approximation for algebraic groups*, Thin groups and superstrong approximation, MSRI Publications **61**, 2013.