18.785 Number Theory **Fall 2015**
Problem Set #1 **Due: 09/21/2015**

Description

These problems are related to the material covered in Lectures 1-2. Your solutions are to be written up in latex (you can use the latex source for the problem set as a template) and submitted as a pdf-file via e-mail to `drew@math.mit.edu` by 5pm on the due date. Collaboration is permitted, but you must identify your collaborators, and any references you consulted. If there are none, write **Sources consulted: none** at the top of your problem set. The first person to spot each non-trivial typo/error in any of the problem sets or lecture notes will receive 1-5 points of extra credit.

Instructions: First solve the warm up problems on scratch paper or a chalk board. Then pick *three* of problems 1–4 to solve and write up your answers in latex. Finally, be sure to complete problem 5 — this is a short survey that will help to shape future problem sets and lectures.

Problem 0.

These are warm up problems that do not need to be written up or turned in (if you are unsure about your answer and want the grader to check your reasoning, you may turn in a solution, but please do not burden the grader gratuitously). These should not take long and are simply provided to help you check your understanding.

(a) Prove that an absolute value $|\cdot|$ on a field k is nonarchimedean if and only if $|n| \le 1$ for every $n \in \mathbb{Z}_{>0}$ (view n as an element of k via the canonical embedding $\mathbb{Z} \to k$).

(b) Prove the nonarchimedean "triangle equality": if $|\cdot|$ is a nonarchimedean absolute value on a field k and $|x| \ne |y|$ then $|x+y| = \max(|x|,|y|)$.

(c) Find a monic polynomial $f \in \mathbb{Z}[x]$ with $\sqrt{2}+\sqrt{3}$ as a root.

Problem 1. Absolute values on $\mathbb{Q}$ (33 points)

(a) Prove Ostrowski's Theorem: every nontrivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime $p \le \infty$ (hint: use warm up problem 2).

(b) Prove the product formula for $\mathbb{Q}$: show that $\prod_{p\le\infty} |x|_p = 1$ for all $x \in \mathbb{Q}^\times$.

Problem 2. Absolute values on $\mathbb{F}_q(t)$ (33 points)

For each prime $\pi \in \mathbb{F}_q[t]$ and any nonzero $f \in \mathbb{F}_q[t]$, let $v_\pi(f)$ be the largest integer for which $\pi^n | f$, equivalently, the largest n for which $f \in (\pi^n)$. For each $f/g \in \mathbb{F}_q(t)^\times$ in lowest terms (meaning f and g have no common factor in $\mathbb{F}_q[t]$), define

$$v_\pi(f/g) := v_\pi(f) - v_\pi(g),$$

and let $v_\pi(0) := \infty$; also define $\deg 0 := -\infty$ and $\deg(f/g) := \deg f - \deg g$.

(a) For each prime $\pi \in \mathbb{F}_q[t]$, define $|r|_\pi = (q^{\deg \pi})^{-v_\pi(r)}$ for all $r \in \mathbb{F}_q(t)$. Show that $|\cdot|_\pi$ is a nonarchimedean absolute value on $\mathbb{F}_q(t)$.

(b) Define $|r|_\infty := q^{\deg r}$, for all $r \in \mathbb{F}_q(t)$. Prove that $|\cdot|_\infty$ is a nonarchimedean absolute value on $\mathbb{F}_q(t)$.

(c) Determine the residue field of $\mathbb{F}_q(t)$ with respect to $|\cdot|_\pi$.

(d) Describe the local ring $R = \{x \in \mathbb{F}_q(t) : |x|_\infty \le 1\}$ and its unique maximal ideal $\mathfrak{m}$. Then determine the residue field of $\mathbb{F}_q(t)$ with respect to $|\cdot|_\infty$.

(e) Prove Ostrowski's theorem for $\mathbb{F}_q(t)$: every nontrivial absolute value on $\mathbb{F}_q(t)$ is equivalent to $|\cdot|_\infty$ or $|\cdot|_\pi$ for some prime $\pi \in \mathbb{F}_q[t]$.

More precisely, show that if $\|\cdot\|$ is a nontrivial absolute value on $\mathbb{F}_q(t)$, either $\|t\| > 1$ and $\|\cdot\| \sim ||_\infty$, or $\|t\| \le 1$ and $\|\cdot\| \sim |\cdot|_\pi$ for some prime $\pi \in \mathbb{F}_q[t]$.

In view of (e) we now regard ∞ as a "prime" of $\mathbb{F}_q(t)$ and let π range over both monic irreducible polynomials in $\mathbb{F}_q[t]$ and ∞.

(f) Prove the product formula for $\mathbb{F}_q(t)$: show that $\prod_\pi |r|_\pi = 1$ for every $r \in \mathbb{F}_q(t)^\times$.

Problem 3. Quadratic fields (33 points)

Let $K = \mathbb{Q}(\sqrt{d})$ with $d \ne 1$ squarefree, and let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}_K$ that does not divide $(2d)$.

(a) Give explicit generators for $\mathcal{O}_K$ as a $\mathbb{Z}$-lattice.

(b) Determine the index of the order $\mathbb{Z}[\sqrt{d}]$ in $\mathcal{O}_K$ as a function of d.

(c) Show that $\mathfrak{p}$ can be written in the form (p, α), with $(p) = \mathfrak{p} \cap \mathbb{Z}$ and $\alpha \in \mathcal{O}_K$.

(d) Show that $\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_q$ where $q = [\mathcal{O}_K : \mathfrak{p}]$ is either p or p^2, with $\mathfrak{p} = (p, \alpha)$. Give an explicit criterion in terms of p and d for when the two cases occur.

Problem 4. Dedekind domains (33 points)

Let A be a Dedekind domain with fraction field K.

(a) Describe all nonzero A-submodules of K.

(b) Describe all subrings of K containing A.

Hint: first think about the case where A is a DVR.

Problem 5. Survey (1 point)

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
9/10	Absolute values				
9/15	Dedekind domains				
9/17	Fractional ideals				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.