25 Global class field theory, the Chebotarev density theorem

25.1 Global fields

Recall that a global field is a field with a product formula whose completion at any nontrivial absolute value is a local field. As proved on Problem Set 7, every such field is one of:

- *number field*: finite extension of $\mathbb{Q}$ (characteristic 0, has an archimedean place);
- *global function field*: finite extension of $\mathbb{F}_p(t)$ (characteristic p, no archimedean places).

An equivalent characterization of a global function field is that it is the function field of a smooth projective (geometrically integral) curve over a finite field.

In Lecture 22 we defined the *adele ring* $\mathbb{A}_K$ of a global field K as the restricted product

$$\mathbb{A}_K := \prod(K_v, \mathcal{O}_v) = \big\{(a_v) \in \prod K_v : a_v \in \mathcal{O}_v \text{ for almost all } v\big\},$$

over all of its places (equivalence classes of absolute values) v; here K_v denotes the completion of K at v (a local field), and $\mathcal{O}_v$ is the valuation ring of K_v if v is nonarchimedean, and $\mathcal{O}_v = K_v$ otherwise. As a topological ring, $\mathbb{A}_K$ is locally compact and Hausdorff. The field K is canonically embedded in $\mathbb{A}_K$ via the diagonal map $x \mapsto (x, x, x, \ldots)$ whose image is discrete, closed, and cocompact; see Theorem 22.12.

In Lecture 23 we defined the idele group

$$\mathbb{I}_K := \prod(K_v^\times, \mathcal{O}_v^\times) = \big\{(a_v) \in \prod K_v^\times : a_v \in \mathcal{O}_v^\times \text{ for almost all } v\big\},$$

which coincides with the unit group of $\mathbb{A}_K$ as a set but not as a topological space (the restricted product topology ensures that $a \mapsto a^{-1}$ is continuous, which is not true of the subspace topology). As a topological group, $\mathbb{I}_K$ is locally compact and Hausdorff. The multiplicative group $K^\times$ is canonically embedded in $\mathbb{I}_K$ via the diagonal map $x \mapsto (x, x, x, \ldots)$, and the *idele class group* is the quotient $C_K := \mathbb{I}_K/K^\times$ (which we recall is not compact; to get a compact quotient one must restrict to the norm-1 subgroup $\mathbb{I}_K^1 := \{a \in \mathbb{I}_K : \|a\| = 1\}$).

25.2 The idele norm

Recall from Lecture 23 that the idele group $\mathbb{I}_K$ surjects onto the ideal group $\mathcal{I}_K$ via

$$a \mapsto \prod \mathfrak{p}^{v_\mathfrak{p}(a)},$$

and we have the following commutative diagram of exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & K^\times \arrow[r] \arrow[d, two heads] & \mathbb{I}_K \arrow[r] \arrow[d, two heads] & C_k \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \mathcal{P}_K \arrow[r] & \mathcal{I}_K \arrow[r] & \mathrm{Cl}_k \arrow[r] & 1
\end{tikzcd}$$

Definition 25.1. Let L/K is a finite separable extension of global fields. The *idele norm* $\mathrm{N}_{L/K} \colon \mathbb{I}_L \to \mathbb{I}_K$ is defined by $\mathrm{N}_{L/K}(b_w) = (a_v)$, where each

$$a_v := \prod_{w|v} \mathrm{N}_{L_w/K_v}(b_w)$$

is a product over places w of L that extend the place v of K and $\mathrm{N}_{L_w/K_v} \colon L_w \to K_v$ is the field norm of the corresponding extension of local fields L_w/K_v.

It follows from Corollary 11.5 and Remark 11.6 that the idele norm $\mathrm{N}_{L/K}\colon \mathbb{I}_L \to \mathbb{I}_K$ agrees with the field norm $\mathrm{N}_{L/K}\colon L^\times \to K^\times$ on the subgroup of principal ideles $L^\times \subseteq \mathbb{I}_L$. The field norm is also compatible with the ideal norm $\mathrm{N}_{L/K}\colon \mathcal{I}_L \to \mathcal{I}_K$ (see Proposition 6.6), and we thus obtain the following commutative diagram

$$\begin{tikzcd} L^\times \arrow[r] \arrow[d, "\mathrm{N}_{L/K}"] & \mathbb{I}_L \arrow[r] \arrow[d, "\mathrm{N}_{L/K}"] & \mathcal{I}_L \arrow[d, "\mathrm{N}_{L/K}"] \\ K^\times \arrow[r] & \mathbb{I}_K \arrow[r] & \mathcal{I}_K \end{tikzcd}$$

The image of $L^\times$ in $\mathbb{I}_L$ under the composition of the maps on the top row is precisely the group $\mathcal{P}_L$ of principal ideals, and the image of $K^\times$ in $\mathbb{I}_K$ is similarly $\mathcal{P}_K$. Taking quotients yields induced norm maps on the idele and ideal class groups that make the following diagram commute:

$$\begin{tikzcd} C_L \arrow[r, two heads] \arrow[d, "\mathrm{N}_{L/K}"] & \mathrm{Cl}_L \arrow[d, "\mathrm{N}_{L/K}"] \\ C_K \arrow[r, two heads] & \mathrm{Cl}_K \end{tikzcd}$$

25.3 The global Artin homomorphism

Let K be a global field and let K^{ab} be its maximal abelian extension inside some fixed separable closure K^{sep}. We are going to use the local Artin homomorphisms introduced in the previous lecture to construct the *global Artin homomorphism*

$$\theta_K\colon \mathbb{I}_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K).$$

For each place v of K we embed the local field K_v into the idele group $\mathbb{I}_K$ via the map

$$\begin{aligned} K_v^\times &\hookrightarrow \mathbb{I}_K \\ \alpha &\mapsto (1,1,\ldots,1,\alpha,1,1,\ldots), \end{aligned}$$

whose image intersects $K^\times \subseteq \mathbb{I}_K$ trivially. This embedding is compatible with the idele norm in the following sense: if L/K is any finite separable extension and w is a place of L that extends the place v of K then the diagram

$$\begin{tikzcd} L_w^\times \arrow[r, "\mathrm{N}_{L_w/K_v}"] \arrow[d, hook] & K_v^\times \arrow[d, hook] \\ \mathbb{I}_L \arrow[r, "\mathrm{N}_{L/K}"] & \mathbb{I}_K \end{tikzcd}$$

commutes.

For each of the local fields K_v let K_v^{ab} be the maximal abelian extension of K_v inside some fixed separable closure of K_v^{sep}. Let

$$\theta_{K_v}\colon K_v^\times \to \mathrm{Gal}(K_v^{\mathrm{ab}}/K_v)$$

be the corresponding local Artin homomorphism. Recall from Theorem 24.2 that for each finite abelian extension L_w/K_v the map $\theta_{L_w/K_v}\colon K_v^\times \to \mathrm{Gal}(L_w/K_v)$ obtained by composing θ_{K_v} with the quotient map $\mathrm{Gal}(K_v^{\mathrm{ab}}/K) \to \mathrm{Gal}(L_w/K_v)$ induces an isomorphism

$K_v^\times/\mathrm{N}_{L_w/K_v}(L_w^\times) \to \mathrm{Gal}(K_v^{\mathrm{ab}}/K)$, and taking profinite completions yields an isomorphism of profinite groups

$$\widehat{\theta}_{K_v}\colon \widehat{K_v^\times} \xrightarrow{\sim} \mathrm{Gal}(K_v^{\mathrm{ab}}/K_v).$$

Let L/K be a finite abelian extension in K^{ab} and let v be a place of K. For each place w of L extending v we embed $\mathrm{Gal}(L_w/K_v)$ in $\mathrm{Gal}(L/K)$ as follows:

- if v is archimedean then either $L_w \simeq K_v$ and we identify $\mathrm{Gal}(L_w/K_v)$ with the trivial subgroup of $\mathrm{Gal}(L/K)$, or $L_w/K_v \simeq \mathbb{C}/\mathbb{R}$ and we identify $\mathrm{Gal}(L_w/K_v)$ with the subgroup of $\mathrm{Gal}(L/K)$ generated by complex conjugation (which must be nontrivial).
- if v is nonarchimedean, let $\mathfrak{q}$ be the prime of L corresponding to the place w and identify $\mathrm{Gal}(L_w/K_v)$ with the decomposition group $D_\mathfrak{q} \subseteq \mathrm{Gal}(L/K)$ via the isomorphism given by part (6) of Theorem 11.4.

We now observe that this embedding is the same for every place w of L extending v: this is obvious in the archimedean case, and in the nonarchimedean case the decomposition groups $D_\mathfrak{q}$ are all conjugate subgroups of $\mathrm{Gal}(L/K)$ and must coincide because $\mathrm{Gal}(L/K)$ is abelian.

For each place v, let L_v be a completion of L at a place $w|v$ and define

$$\begin{aligned}\theta_{L/K}\colon \mathbb{I}_K &\to \mathrm{Gal}(L/K)\\ (a_v) &\mapsto \prod_v \theta_{L_v/K_v}(a_v),\end{aligned}$$

where the product takes place in $\mathrm{Gal}(L/K)$ via the embeddings $\mathrm{Gal}(L_v/K_v) \subseteq \mathrm{Gal}(L/K)$. This is a finite product because almost for almost all v we have $a_v \in \mathcal{O}_v^\times$ and v unramified in L, in which case $\theta_{L_v/K_v}(a_v)$ is necessarily trivial. It is clear that $\theta_{L/K}$ is a homomorphism, since each θ_{L_w/K_v} is, and $\theta_{L/K}$ is continuous, since the inverse image of any subgroup of the finite group $\mathrm{Gal}(L/K)$ is clearly an open subgroup of $\mathbb{I}_K$ (because almost all of its projections to $K_v^\times$ contain $\mathcal{O}_v^\times$).

If $L_1 \subseteq L_2$ are two finite abelian extensions of K, then then $\theta_{L_2/K}(x)_{|L_1} = \theta_{L_1/K}(x)$ for all $x \in \mathbb{I}_K$; thus the $\theta_{L/K}$ form a compatible system of homomorphisms from $\mathbb{I}_K$ to the inverse limit $\varprojlim L/K$, where L/K ranges over finite abelian extensions of K in K^{ab} ordered by inclusion, and they thus determine a continuous homomorphism $\mathbb{I}_K \to \varprojlim \mathrm{Gal}(L/K)$.

Definition 25.2. The *global Artin homomorphism* is the continuous homomorphism

$$\theta_K\colon \mathbb{I}_K \to \varprojlim \mathrm{Gal}(L/K) \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

determined by the compatible system of homomorphisms $\theta_{L/K}\colon \mathbb{I}_K \to \mathrm{Gal}(L/K)$.

The isomorphism $\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \varprojlim \mathrm{Gal}(L/K)$ is the natural isomorphism between a Galois group and its profinite completion with respect to the Krull topology (see Theorem 23.22) and is thus canonical, as is the global Artin homomorphism θ_K.

Proposition 25.3. *Let K be global field. The global Artin homomorphism θ_K is the unique continuous homomorphism $\mathbb{I}_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ with the property that for every finite abelian extension L/K in K^{ab} and place w of L lying over a place v of K the diagram*

$$\begin{tikzcd} K_v^\times \arrow[r, "\theta_{L_w/K_v}"] \arrow[d, hook] & \mathrm{Gal}(L_w/K_v) \arrow[d, hook] \\ \mathbb{I}_K \arrow[r, "\theta_{L/K}"] & \mathrm{Gal}(L/K) \end{tikzcd}$$

commutes, where $\theta_{L/K}(x) := \theta_K(x)_{|_L}$.

Proof. That θ_K has this property follows directly from its construction. Now suppose $\theta'_K \colon \mathbb{I}_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ has the same property. The idele group $\mathbb{I}_K$ is generated by the images of the embeddings $K_v^\times$, so if θ_K and θ'_K are not identical, then they disagree at a point $a := (1, 1, \ldots, 1, a_v, 1, 1, \ldots)$ in the image of one the embeddings $K_v \hookrightarrow \mathbb{I}_K$. We must have $\theta_{L/K}(a) = \phi(\theta_{L_w/K_v}(a_v)) = \theta'_{L/K}(a)$ for every finite abelian extension L/K in K^{ab} and every place w of L extending v, where ϕ is the embedding on the RHS of the commutative diagram. This implies $\theta_K(a) = \theta'_K(a)$, since the image of a under any homomorphism to $\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \varprojlim \mathrm{Gal}(L/K)$ is determined by its images in the $\mathrm{Gal}(L/K)$. $\square$

25.4 The main theorems of global class field theory

In the global version of Artin reciprocity, the idele class group $C_K := \mathbb{I}_K/K^\times$ plays the role that the multiplicative group $K_v^\times$ plays in local Artin reciprocity (Theorem 24.2).

Theorem 25.4 (Global Artin Reciprocity)**.** *Let K be a global field. The kernel of the global Artin homomorphism θ_K contains $K^\times$, thus it induces a continuous homomorphism*

$$\theta_K \colon C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K).$$

For every finite abelian extension L/K in K^{ab} the homomorphism

$$\theta_{L/K} \colon C_K \to \mathrm{Gal}(L/K)$$

obtained by composing θ_K with the quotient map $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(L/K)$ is surjective with kernel $\mathrm{N}_{L/K}(C_L)$ and thus induces an isomorphism $C_K/\mathrm{N}_{L/K}(C_L) \simeq \mathrm{Gal}(L/K)$.

Remark 25.5. When K is a number field, θ_K is surjective but not injective; its kernel is the connected component of the identity in C_K. When K is a global function field, θ_K is injective but not surjective; its image is consists of automorphisms $\sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ corresponding to integer powers of the Frobenius automorphism of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, where k is the constant field of K (this is precisely the dense image of $\mathbb{Z}$ in $\widehat{\mathbb{Z}} \simeq \mathrm{Gal}(k^{\mathrm{sep}}/k)$).

We also have a global existence theorem.

Theorem 25.6 (Global Existence Theorem)**.** *Let K be a global field. For every finite index open subgroup H of C_K there is a unique finite abelian extension L/K inside K^{ab} for which $\mathrm{N}_{L/K}(C_L) = H$.*

As with the local Artin homomorphism, taking profinite completions yields an isomorphism that allows us to summarize global class field theory in one statement.

Theorem 25.7 (Main theorem of global class field theory)**.** *Let K be a global field. The global Artin homomorphism θ_K induces a canonical isomorphism*

$$\widehat{\theta}_K \colon \widehat{C_K} \xrightarrow{\sim} \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

of profinite groups.

We thus have inclusion preserving bijections

$$\{\text{ closed subgroups } H \text{ of } C_K\} \longleftrightarrow \{\text{ abelian extensions } L/K \text{ in } K^{\mathrm{ab}}\}$$
$$\{\text{ finite index open subgroups } H \text{ of } C_K\} \longleftrightarrow \{\text{ finite abelian extensions } L/K \text{ in } K^{\mathrm{ab}}\}$$

and corresponding isomorphisms $C_K/H \simeq \mathrm{Gal}(L/K)$, where $H = \mathrm{N}_{L/K}(C_L)$. We also note that the global Artin homomorphism is *functorial* in the following sense.

Theorem 25.8 (Functoriality)**.** *Let K be a global field and let L/K be any finite separable extension (not necessarily abelian). Then the following diagram commutes*

$$\begin{tikzcd} C_L \arrow[r, "\theta_L"] \arrow[d, "N_{L/K}"] & \mathrm{Gal}(L^{\mathrm{ab}}/L) \arrow[d, "\mathrm{res}"] \\ C_K \arrow[r, "\theta_K"] & \mathrm{Gal}(K^{\mathrm{ab}}/K). \end{tikzcd}$$

25.5 Relation to ideal-theoretic version of global class field theory

Let K be a global field and let $\mathfrak{m}$ be a modulus for K, which we recall is a formal product $\mathfrak{m} = \prod v^{e_v}$ over the places v of K with $e_v \geq 0$, where $e_v \leq 1$ when v is a real archimedean place and $e_v = 0$ when v is a complex archimedean place (see Definition 20.1). For each place v of K we define the multiplicative group

$$U_K^{\mathfrak{m}}(v) := \begin{cases} \mathcal{O}_v^\times & \text{if } v \nmid \mathfrak{m}, \text{ where } \mathcal{O}_v^\times := K_v^\times \text{ when } v \text{ is infinite}), \\ \mathbb{R}_{>0} & \text{if } v|\mathfrak{m} \text{ is real, where } \mathbb{R}_{>0} \subseteq \mathbb{R}^\times \simeq K_v^\times, \\ 1 + \mathfrak{p}^{e_v} & \text{if } v|\mathfrak{m} \text{ is finite, where } \mathfrak{p} = \{x \in \mathcal{O}_v : |x|_v < 1\}. \end{cases}$$

We then let $U_K^{\mathfrak{m}} := \prod_v U_K^{\mathfrak{m}}(v) \subseteq \mathbb{I}_K$ be the corresponding open subgroup of $\mathbb{I}_K$. The image $\overline{U}_K^{\mathfrak{m}}$ of $U_K^{\mathfrak{m}}$ in the idele class group $C_K = \mathbb{I}_K/K^\times$ is an open subgroup with finite index. The idelic version of a ray class group is the quotient

$$C_K^{\mathfrak{m}} := \mathbb{I}_K/(U_K^{\mathfrak{m}} \cdot K^\times) = C_K/\overline{U}_K^{\mathfrak{m}},$$

and we have isomorphisms

$$C_K^{\mathfrak{m}} \simeq \mathrm{Cl}_K^{\mathfrak{m}} \simeq \mathrm{Gal}(K(\mathfrak{m})/K),$$

where $\mathrm{Cl}_K^{\mathfrak{m}}$ is the ray class group for the modulus $\mathfrak{m}$ (see Definition 20.2), and $K(\mathfrak{m})$ is the corresponding *ray class field*, which we can now define as the finite abelian extension L/K for which $\mathrm{N}_{L/K}(C_L) = \overline{U}_K^{\mathfrak{m}}$, whose existence is guaranteed by Theorem 25.6.

If L/K is any finite abelian extension, then $\mathrm{N}_{L/K}(C_L)$ contains $\overline{U}_L^{\mathfrak{m}}$ for some modulus $\mathfrak{m}$; this follows from the fact that the groups $\overline{U}_L^{\mathfrak{m}}$ form a fundamental system of open neighborhoods of the identity. Indeed, the conductor of the extension L/K (see Definition 21.15) is precisely the minimal modulus $\mathfrak{m}$ for which this is true. It follows that every abelian extension L/K lies in a ray class field $K(\mathfrak{m})$ and has Galois group isomorphic to a quotient of a ray class group $C_K^{\mathfrak{m}}$.

25.6 The Chebotarev density theorem

We now give a proof of the Chebotarev density theorem, a generalization of the Frobenius density theorem you proved on Problem Set 10. Recall from Lecture 17 (and Problem Set 9) that if S is a set of primes of a global field K, the *Dirichlet density* of S is defined by

$$d(S) := \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in S} \mathrm{N}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}} \mathrm{N}(\mathfrak{p})^{-s}} = \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in S} \mathrm{N}(\mathfrak{p})^{-s}}{\log \frac{1}{s-1}},$$

whenever this limit exists. As you proved on Problem Set 9, if S has a natural density then it has a Dirichlet density and the two coincide (and similarly for polar density). A subset C of a group is said to be *stable under conjugation* if $\sigma\tau\sigma^{-1} \in C$ for all $\sigma \in G$ and $\tau \in C$.

Theorem 25.9 (Chebotarev density theorem)**.** *Let L/K be a finite Galois extension of number fields with Galois group $G := \mathrm{Gal}(L/K)$. Let $C \subseteq G$ be stable under conjugation, and let S be the set of primes $\mathfrak{p}$ of K unramified in L with $\mathrm{Frob}_{\mathfrak{p}} \subseteq C$. Then $d(S) = \#C/\#G$.*

Note that G is not assumed to be abelian, so $\mathrm{Frob}_{\mathfrak{p}}$ is a conjugacy class, not an element. However, the main difficulty in proving the Chebotarev density theorem (and the only place where class field theory is actually needed) occurs when G is abelian. The main result we need is the generalization of Dirichlet's theorem on primes in arithmetic progressions to number fields, which we proved in Lecture 21, subject to the existence of ray class fields, which we now assume.

Proposition 25.10. *Let $\mathfrak{m}$ be a modulus for a number field K and let $\mathrm{Cl}_K^{\mathfrak{m}}$ be the corresponding ray class group. For every ray class $c \in \mathrm{Cl}_K^{\mathfrak{m}}$ the Dirichlet density of the set of primes $\mathfrak{p}$ of K that are prime to $\mathfrak{m}$ and lie in c is $1/\#\mathrm{Cl}_K^{\mathfrak{m}}$.*

Proof. This follows from Theorem 25.6, which guarantees that the ray class field $K(\mathfrak{m})$ exists, and Theorem 21.11, Corollary 21.12, and Corollary 21.14, from Lecture 21. □

Corollary 25.11. *Let L/K be an abelian extension of number fields with Galois group G. For every $\sigma \in G$ the Dirichlet density of the set S of unramified primes $\mathfrak{p}$ of K for which $\mathrm{Frob}_{\mathfrak{p}} = \{\sigma\}$ is $1/\#G$.*

Proof. Let $\mathfrak{m} = \mathrm{cond}(L/K)$ be the conductor of the extension L/K; then L is a subfield of the ray class field $K(\mathfrak{m})$ and $\mathrm{Gal}(L/K) \simeq \mathrm{Cl}_K^{\mathfrak{m}}/H$ for some subgroup H of the ray class group. For each unramified prime $\mathfrak{p}$ of K we have $\mathrm{Frob}_{\mathfrak{p}} = \{\sigma\}$ if and only if $\mathfrak{p}$ lies in one of the ray classes contained in the coset of H in $\mathrm{Cl}_K^{\mathfrak{m}}/H$ corresponding to σ. The Dirichlet density of the set of primes in each ray class is $1/\#\mathrm{Cl}_K^{\mathfrak{m}}$, by Proposition 25.10, and there are $\#H$ ray classes in each coset of H; thus $d(S) = \#H/\#\mathrm{Cl}_K^{\mathfrak{m}} = 1/\#G$. □

Proof of the Chebotarev density theorem. We first note that it suffices to prove the theorem in the case that C is a single conjugacy class, since we can always partition C into conjugacy classes and sum Dirichlet densities (as proved on Problem Set 9). If L/K is abelian then $\#C = 1$ and the theorem follows from Corollary 25.11.

For the general case, let σ be a representative of the conjugacy class C, let $H = \langle\sigma\rangle$ be the subgroup of G it generates, and let $F = L^H$ be the corresponding fixed field. Let T be the set of primes $\mathfrak{q}$ of F that are unramified in L for which $\mathrm{Frob}_{\mathfrak{q}} = \sigma$; the extension L/F is abelian with Galois group H, so $d(T) = 1/\#H$ follows from the abelian case. Restricting to degree-1 primes (primes whose residue field has prime order) does not change Dirichlet

densities (as you proved on Problem Set 9), so we now restrict S and T to their subsets of degree-1 primes.

We claim that for each prime $\mathfrak{p} \in S$ there are exactly $\#G/(\#H\#C)$ primes $\mathfrak{q} \in T$ that lie above $\mathfrak{p}$. Assuming the claim, we have

$$\sum_{\mathfrak{p}\in S} \mathrm{N}(\mathfrak{p})^{-s} = \frac{\#H\#C}{\#G} \sum_{\mathfrak{q}\in T} \mathrm{N}(\mathfrak{q})^{-s},$$

since $\mathrm{N}(\mathfrak{q}) = \mathrm{N}(\mathfrak{p})$ for each degree-1 prime $\mathfrak{q}$ lying above a degree-1 prime $\mathfrak{p}$; it follows that

$$d(S) = \frac{\#H\#C}{\#G} d(T) = \frac{\#C}{\#G}$$

as desired.

We now prove the claim. Let U be the set of primes $\mathfrak{r}$ of L for which $\mathfrak{r} \cap K = \mathfrak{p} \in S$ and $\mathrm{Frob}_{\mathfrak{r}} = \sigma$. For each $\mathfrak{r} \in U$, if we put $\mathfrak{q} := \mathfrak{r} \cap F$ then $\mathrm{Frob}_{\mathfrak{q}} = \sigma$, and since σ fixes F it acts trivially on $\mathbb{F}_{\mathfrak{q}} := \mathcal{O}_F/\mathfrak{q}$, so the residue field extension $\mathbb{F}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{p}}$ is trivial and $\mathfrak{q} \in T$. On the other hand, $\mathrm{Gal}(L/F) = \langle\sigma\rangle = H$, so the residue field extension $\mathbb{F}_{\mathfrak{r}}/\mathbb{F}_{\mathfrak{q}}$ has degree $\#H$, and this implies that $\mathfrak{r}$ is the only prime of L above $\mathfrak{q}$. Conversely, for each $\mathfrak{q} \in F$, every prime $\mathfrak{r}$ of L above $\mathfrak{q}$ must have $\mathrm{Frob}_r = \sigma$ (the $\mathrm{Frob}_{\mathfrak{r}}$ are conjugate in $H = \mathrm{Gal}(L/F)$ and must be equal since H is abelian), hence lie in U, and have residue degree $[\mathbb{F}_{\mathfrak{r}} : \mathbb{F}_{\mathfrak{q}}] = \#H$, hence be the unique prime of L above $\mathfrak{q}$.

The sets U and T are thus in bijection, so to count the primes $\mathfrak{q} \in T$ that lie above some prime $\mathfrak{p} \in S$ it suffices to count the primes $\mathfrak{r} \in U$ that lie above $\mathfrak{p}$. The set X of primes $\mathfrak{r}$ of L that lie above $\mathfrak{p}$ has cardinality $\#G/\#H$, since the primes $\#r$ are all unramified and have residue degree $\#H$. The transitive action of G on X partitions it into $\#C$ orbits corresponding to the conjugates of σ, each of which has size $\#G/(\#H\#C)$. Each orbit corresponds to a possible value of $\mathrm{Frob}_{\mathfrak{r}}$, all of which are conjugate to σ and exactly one of which is equal to σ; this orbit is the set of primes $\mathfrak{r}$ of L above $\mathfrak{p}$ that lie in U and has cardinality $\#G/(\#H\#C)$, which proves the claim and completes the proof. $\square$

Remark 25.12. The Chebotarev density theorem holds for any global field; the generalization to function fields was originally proved by Reichardt [3]; see [2] for a modern proof (and in fact a stronger result). In the case of number fields (but not function fields!) Chebotarev's theorem also holds for natural density. This follows from results of Hecke [1] that actually predate Chebotarev's work; Hecke showed that the primes lying in any particular ray class (element of the ray class group) have a natural density.

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