16 The functional equation

In the course of proving the Prime Number Theorem we showed that the Riemann zeta function $\zeta(s) := \sum_{n\geq 1} n^{-s}$ has an Euler product and an analytic continuation to the right half-plane $\mathrm{Re}(s) > 0$. We now want to complete the picture by deriving a *functional equation* that relates the values of $\zeta(s)$ to values of $\zeta(1-s)$. This will also allow us to extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ (holomorphic except for a simple pole at $s = 1$). Thus $\zeta(s)$ satisfies the three key properties that we would like any zeta function (or L-series) to have:

- an Euler product;
- an analytic continuation;
- a functional equation.

16.1 Fourier transforms and Poisson summation

A key ingredient to the functional equation is the Poisson summation formula, a tool from functional analysis that we now recall.

Definition 16.1. A *Schwartz function* on $\mathbb{R}$ is a complex-valued $\mathbb{C}^\infty$-function $f\colon \mathbb{R} \to \mathbb{C}$ that decays rapidly to zero; more precisely, we require that for all $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\sup_{x\in\mathbb{R}} \left| x^m f^{(n)}(x) \right| < \infty,$$

where $f^{(n)}$ denotes the nth derivative of f. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ of all Schwartz functions on $\mathbb{R}$ is a $\mathbb{C}$-vector space (and also a complete topological space, but its topology will not concern us here). It is closed under differentiation and products, and also under *convolution*: for any $f, g \in \mathcal{S}(\mathbb{R})$ the function

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)dy$$

is also in $\mathcal{S}(\mathbb{R})$.

Examples of Schwartz functions include all compactly supported functions C^∞ functions, as well as the Gaussian $g(x) := e^{-\pi x^2}$, which is the main case of interest to us.

Definition 16.2. The *Fourier transform* of a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ is the function

$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e^{-2\pi i x y}dx,$$

which is also a Schwartz function. The Fourier transform is an invertible linear operator on the vector space $\mathcal{S}(\mathbb{R})$; the inverse transform of $\hat{f}(y)$ is

$$f(x) := \int_{\mathbb{R}} \hat{f}(y)e^{+2\pi i x y}dy.$$

The Fourier transform changes convolutions into products, and vice versa. We have

$$\widehat{f * g} = \hat{f}\hat{g} \qquad \text{and} \qquad \widehat{fg} = \hat{f} * \hat{g},$$

for all $f, g \in \mathcal{S}(\mathbb{R})$.

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Theorem 16.3 (Poisson Summation Formula)**.** *For all $f \in \mathcal{S}(\mathbb{R})$ we have the identity*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Proof. We first note that both sums are well defined; the rapid decay property of Schwartz functions guarantees absolute convergence. Let $F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$. Then F is a periodic C^∞-function, so it has a Fourier series expansion

$$F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x},$$

with Fourier coefficients

$$c_n = \int_0^1 F(x)e^{-2\pi i n t}dt = \int_0^1 \sum_{m\in\mathbb{Z}} f(x+m)e^{-2\pi i n y}dy = \int_{\mathbb{R}} f(x)e^{-2\pi i n y}dy = \hat{f}(n).$$

We then note that

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{x\to 0} F(x) = \lim_{x\to 0}\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} \lim_{x\to 0} \hat{f}(n)e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where we have used $f \in \mathcal{S}(\mathbb{R})$ to justify interchanging the limit and sum (alternatively, one can view the limit as a uniformly converging sequence of functions). ☐

We now note that the Gaussian function $g(x) := e^{-\pi x^2}$ is its own Fourier transform.

Lemma 16.4. *Let $g(x) := e^{-\pi x^2}$. Then $\hat{g}(y) = g(y)$.*

Proof. We have

$$\begin{aligned}
\hat{g}(y) &= \int_{-\infty}^{+\infty} e^{-\pi x^2}e^{-2\pi i x y}dx = \int_{-\infty}^{+\infty} e^{-\pi(x^2+2ixy+y^2-y^2)}dx\\
&= e^{-\pi y^2}\int_{-\infty}^{+\infty} e^{-\pi(x+iy)^2}dx = e^{-\pi y^2}\int_{-\infty+iy}^{+\infty+iy} e^{-\pi(x+iy)^2}dx\\
&= e^{-\pi y^2}\int_{-\infty}^{+\infty} e^{-\pi t^2}dt = e^{-\pi y^2} = g(y).
\end{aligned}$$

We used a contour integral of the holomorphic function $f(x+iy) = e^{-\pi(x+iy)^2}$ along the rectangular contour $-r \to r \to r+i \to -r+i \to -r$ with $r \to \infty$ to shift the integral up by i in the second line: the integral along the vertical sides vanishes as $r \to \infty$, so the contributions form the horizontal sides must be equal and opposite. We used the change of variable $t = x + iy$ to get the third line, and note that $\int_{-\infty}^{+\infty} e^{-\pi t^2}dx = 1$, because $e^{-\pi t^2}$ is a probability distribution (or insert your favorite proof of this fact here). ☐

Corollary 16.5. *For any $a \in \mathbb{R}^\times$, if $G_a(x) := g(x/\sqrt{a})$ then $\widehat{G}_a(y) = \sqrt{a}g(y\sqrt{a})$.*

Proof. Proceeding as in the first line of the lemma and substituting $x \to \sqrt{a}x$ yields

$$\begin{aligned}
\widehat{G}_a(y) &= \int_{-\infty}^{+\infty} e^{-\pi x^2/a}e^{-2\pi i x y}dx = \sqrt{a}\int_{-\infty}^{+\infty} e^{-\pi(x^2+2ixy\sqrt{a}+y^2a-y^2a)}dx\\
&= \sqrt{a}e^{-\pi y^2 a}\cdot\int_{-\infty}^{+\infty} e^{-\pi(x+iy\sqrt{a})^2}dx = \sqrt{a}g(y\sqrt{a})\cdot 1.
\end{aligned}$$

☐

16.1.1 Jacobi's theta function

We now define the *theta function*[1]

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}.$$

The sum is absolutely convergent for $\operatorname{Im}\tau > 0$ and thus defines a holomorphic function on the upper half plane. It is easy to see that $\Theta(\tau)$ is periodic modulo 2, that is

$$\Theta(\tau+2) = \Theta(\tau),$$

but it it also satisfies another functional equation.

Lemma 16.6. *For all $y \in \mathbb{R}_{>0}$ we have*

$$\Theta(i/y) = \sqrt{y}\,\Theta(iy)$$

Proof. Plugging $\tau = iy$ into $\Theta(\tau)$ yields

$$\Theta(iy) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}.$$

Applying Corollary 16.5 to $G_y(n) = e^{-\pi n^2/y}$, we have $\widehat{G}_y(n) = \sqrt{y}e^{-\pi n^2 y}$, and Poisson summation (Theorem 16.3) yields

$$\Theta(iy) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{y}}\widehat{G}_y(n) = \frac{1}{\sqrt{y}}\sum_{n\in\mathbb{Z}} G_y(n) = \frac{1}{\sqrt{y}}\Theta\left(i/y\right).$$

The lemma follows. □

16.1.2 Euler's gamma function

You are probably familiar with the gamma function $\Gamma(s)$, which plays a key role in the functional equation of not only the Riemann zeta function but many of the more general zeta functions and L-series we wish to consider. Here we recall some of its analytic properties. We begin with the definition of $\Gamma(s)$ as a Mellin transform.

Definition 16.7. The *Mellin transform* of a function $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ is the complex function defined by

$$\mathcal{M}(f)(s) := \int_0^\infty f(t)t^{s-1}dt,$$

whenever this integral converges. It is holomorphic on $\operatorname{Re} s \in (a,b)$ for any interval (a,b) where the integral $\int_0^\infty |f(t)|t^{\sigma-1}dt$ converges for all $\sigma \in (a,b)$.

Definition 16.8. The *Gamma function*

$$\Gamma(s) := \mathcal{M}(e^{-t})(s) = \int_0^\infty e^{-t}t^{s-1}dt,$$

[1] The function $\Theta(\tau)$ we define here is a special case of one of four parameterized families of theta functions $\Theta_i(z:\tau)$ originally defined by Jacobi for $i = 0,1,2,3$, which play an important role in the theory of elliptic functions and modular forms; in terms of Jacobi's notation, $\Theta(\tau) = \Theta_3(0;\tau)$.

is the Mellin transform of e^{-t}. Since $\int_0^\infty |e^{-t}|t^{\sigma-1}dt$ converges for all $\sigma > 0$, the integral defines a holomorphic function on $\mathrm{Re}(s) > 0$.

Integration by parts yields

$$\Gamma(s) = \frac{t^s e^{-t}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-t}t^s dt = \frac{\Gamma(s+1)}{s},$$

thus $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 (since $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$), and

$$\Gamma(s+1) = s\Gamma(s) \tag{1}$$

for $\mathrm{Re}(s) > 0$. Equation (1) allows us to extend $\Gamma(s)$ to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, and no other poles.

An immediate consequence of (1) is that for integers $n > 0$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1) = n!,$$

thus the gamma function can be viewed as an extension of the factorial function. The gamma function satisfies many useful identities in addition to (1), including the following.

Theorem 16.9 (Euler's Reflection Formula)**.** *We have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

as meromorphic functions on $\mathbb{C}$ with simple poles at each integer $s \in \mathbb{Z}$.

Proof. See [1, §6 Thm. 1.4] □

Example 16.10. Putting $s = \frac{1}{2}$ in the reflection formula yields $\Gamma(\frac{1}{2})^2 = \pi$, so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Corollary 16.11. *The function $\Gamma(s)$ has no zeros on $\mathbb{C}$.*

Proof. Suppose $\Gamma(s_0) = 0$. The RHS of Euler's reflection formula is never zero, since $\sin(\pi s)$ has no poles, so $\Gamma(1-s)$ must have a pole at s_0. Therefore $1 - s_0 \in \mathbb{Z}_{\leq 0}$, equivalently, $s_0 \in \mathbb{Z}_{\geq 1}$, but $\Gamma(s) = (s-1)! \neq 0$ for $s \in \mathbb{Z}_{\geq 1}$. □

16.1.3 Completing the zeta function

Let us now consider the function

$$F(s) := \pi^{-s}\Gamma(s)\zeta(2s),$$

which is a meromorphic on $\mathbb{C}$ and holomorphic on $\mathrm{Re}(s) > 1/2$. We will restrict our attention the this region, in which the sum $\sum_{n\geq 1} n^{-2s}$ defining $\zeta(2s)$ is absolutely convergent.

We have

$$F(s) = \sum_{n\geq 1}(\pi n^2)^{-s}\Gamma(s) = \sum_{n\geq 1}\int_0^\infty (\pi n^2)^{-s}t^{s-1}e^{-t}dt,$$

and the substitution $t = \pi n^2 y$ with $dt = \pi n^2 dy$ yields

$$F(s) = \sum_{n\geq 1}\int_0^\infty (\pi n^2)^{-s}(\pi n^2 y)^{s-1}e^{-\pi n^2 y}\pi n^2 dy = \sum_{n\geq 1}\int_0^\infty y^{s-1}e^{-\pi n^2 y}dy.$$

The sum is absolutely convergent, so by the Fubini-Tonelli theorem, we can swap the sum and the integral to obtain

$$F(s) = \int_0^\infty y^{s-1} \sum_{n\ge 1} e^{-\pi n^2 y} dy.$$

We have $\Theta(iy) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = 1 + 2\sum_{n\ge 1} e^{-\pi n^2 y}$, thus

$$\begin{aligned} F(s) &= \frac{1}{2}\int_0^\infty y^{s-1}(\Theta(iy)-1)dy \\ &= \frac{1}{2}\left(\int_0^1 y^{s-1}\Theta(iy)dy - \frac{1}{s} + \int_1^\infty y^{s-1}(\Theta(iy)-1)dy\right) \end{aligned}$$

We now focus on the first integral. Making the change of variable $t = \frac{1}{y}$ yields

$$\int_0^1 y^{s-1}\Theta(iy)dy = \int_\infty^1 t^{1-s}\Theta(i/t)(-t^{-2})dt = \int_1^\infty t^{-s-1}\Theta(i/t)dt.$$

By Lemma 16.6, $\Theta(i/t) = \sqrt{t}\Theta(it)$, and adding $-\int_1^\infty t^{-s-1/2}dt + \int_1^\infty t^{-s-1/2}dt = 0$ yields

$$\begin{aligned} &= \int_1^\infty t^{-s-1/2}\big(\Theta(it)dt - 1\big)dt + \int_1^\infty t^{-s-1/2}dt \\ &= \int_1^\infty t^{-s-1/2}\big(\Theta(it)dt - 1\big)dt - \frac{2}{1-2s}. \end{aligned}$$

Plugging this back into our equation for $F(s)$ we obtain

$$F(s) = \frac{1}{2}\int_1^\infty \big(y^{s-1} + y^{-s-1/2}\big)\big(\Theta(iy) - 1\big)dy - \frac{1}{2s} - \frac{1}{1-2s}.$$

We now observe that $F(s) = F(\frac{1}{2} - s)$, allowing us to extend $F(s)$ to a meromorphic function on $\mathbb{C}$. Replacing s with $s/2$ leads us to define the *completed zeta function*

$$Z(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

which is meromorphic on $\mathbb{C}$ and satisfies the *functional equation*

$$Z(1-s) = Z(s).$$

It has simple poles at 0 and 1 (and no other poles). The only zeros of $Z(s)$ on $\mathrm{Re}(s) > 0$ are the zeros of $\zeta(s)$, since by Corollary 16.11, the gamma function $\Gamma(s)$ has no zeros (and neither does $\pi^{-s/2}$). Thus the zeros of $Z(s)$ on $\mathbb{C}$ all lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

The functional equation also allows us to extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$. It has no poles other than the simple pole at 1, since $\pi^{-s/2}\Gamma(s)$ has no zeros and the simple pole of $Z(s)$ at 0 corresponds to the simple pole of $\Gamma(s/2)$ at zero. Notice that $\Gamma(s/2)$ has poles at $0, -2, -4, \ldots$, so our extended $\zeta(s)$ must have zeros at $-2, -4, \ldots$ (but not at 0). These are the *trivial zeros* of $\zeta(s)$; all the interesting zeros lie in the critical strip (and under the Riemann hypothesis, on the critical line $\mathrm{Re}(s) = 1/2$, the axis of symmetry in the functional equation).

We can determine the value of $\zeta(0)$ via the functional equation. We know that $\zeta(s)$ has a pole of residue 1 at $s = 1$, thus

$$1 = \lim_{s\to 1^+} (s-1)\zeta(s) = \lim_{s\to 1^+} \frac{(s-1)\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)}{\pi^{-s/2}\Gamma(\frac{s}{2})}.$$

In the limit the denominator on the RHS is 1, since $\Gamma(1/2) = \pi^{1/2}$, and in the numerator we have $\pi^{(s-1)/2} = 1$. Using $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$ to shift the gamma factor in the numerator,

$$1 = \lim_{s\to 1^+} (s-1)\tfrac{2}{1-s}\Gamma\left(\tfrac{3-s}{2}\right)\zeta(0) = -2\Gamma(1)\zeta(0) = -2\zeta(0),$$

thus $\zeta(0) = -1/2$.

If we write out the Euler product for the completed zeta function, we have

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\prod_p (1-p^{-s})^{-1}.$$

One should think of this as a product over the places of the field $\mathbb{Q}$; the leading factor $\Gamma_{\mathbb{R}} := \pi^{-2/s}\Gamma(s/2)$ that distinguishes the completed zeta function $Z(s)$ from $\zeta(s)$ corresponds to the real archimedean place of $\mathbb{Q}$. When we discuss Dedekind zeta functions in a later lecture we will see that there are gamma factors $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ associated to each of the real and complex archimedean places. If we incorporate an additional factor of $\frac{1}{2}s(s-1)$ in $Z(s)$ we can remove the poles at 0 and 1, yielding an entire function $\xi(s)$.

Theorem 16.12 (Analytic Continuation II)**.** *The function*

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is holomorphic on $\mathbb{C}$ and satisfies the functional equation

$$\xi(1-s) = \xi(s).$$

The zeros of $\xi(s)$ all lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

Remark 16.13. It is usually more convenient to just work with $Z(s)$ and deal with the poles rather than making it holomorphic by introducing additional factors; some authors use $\xi(s)$ to denote our $Z(s)$.

References

[1] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton University Press, 2003.