

**Description**

These problems are related to the material covered in Lectures 3-4.

**Instructions:** Solve any combination of Problems 1-5 that sums to 100 points; then complete Problem 6, which is a short survey. Your solutions are to be written up in latex and submitted as a pdf-file with a filename of the form SurnamePset2.pdf via e-mail to drew@math.mit.edu by **11:59 p.m.** on the date due.

Collaboration is permitted/encouraged, but you must identify your collaborators, and any references not listed in the course syllabus. The first to spot each non-trivial typo/error in the problem sets or lecture notes will receive 1-5 points of extra credit.

**Problem 1. Eight ways to Fibonacci (20 points)**

Let  $F_k$  denote the  $k$ th Fibonacci number, defined by:

$$F_1 := 1, \quad F_2 := 1, \quad F_k := F_{k-2} + F_{k-1} \quad (k \geq 2). \quad (1)$$

Analyze the time complexity of computing  $F_k$  modulo an  $n$ -bit prime  $p$  using each of the algorithms listed below. You may assume that  $k > n$ . Your answers should be in the form  $O(f(k, n))$  and as tight as you can make them. Use  $M(n)$  to denote the complexity of multiplying two  $n$ -bit integers.

- (a) Compute the integers  $F_1, F_2, \dots, F_k$  via (1), and then reduce  $F_k \pmod p$ .
- (b) Same as (a), except perform all computations modulo  $p$  (i.e., work in  $\mathbb{F}_p$ ).
- (c) Assume  $p \equiv \pm 1 \pmod 5$ . Compute the roots  $\phi$  and  $\psi$  of  $x^2 - x - 1$  in  $\mathbb{F}_p$  (using a probabilistic root-finding algorithm), and then compute  $F_k = \frac{\phi^k - \psi^k}{\phi - \psi}$  in  $\mathbb{F}_p$ .
- (d) Assume  $p \not\equiv \pm 1 \pmod 5$  and  $p \neq 5$ . Represent  $\mathbb{F}_{p^2}$  as  $\mathbb{F}_p[\phi]/(\phi^2 - \phi - 1)$ , let  $\psi = 1 - \phi$ , and then compute  $F_k = \frac{\phi^k - \psi^k}{\phi - \psi}$  as an element of  $\mathbb{F}_{p^2}$  that actually lies in  $\mathbb{F}_p$ .
- (e) Compute the  $k$ th power of  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$  in  $\text{GL}_2(\mathbb{F}_p)$  and output its upper right entry.
- (f) Assume  $k$  is a power of 2. Use the identities

$$\begin{aligned} F_{2i-1} &= F_i^2 + F_{i-1}^2 \\ F_{2i} &= (2F_{i-1} + F_i)F_i \end{aligned}$$

to compute  $(F_1, F_2), (F_3, F_4), (F_7, F_8), \dots, (F_{k-1}, F_k)$ , working modulo  $p$  throughout.<sup>1</sup>

- (g) Notably absent from the above list is the standard recursive algorithm:

<sup>1</sup>This approach can be generalized to arbitrary  $k$ .

```
def fibonacci(k):
    if k <= 2: return 1
    return fibonacci(k-1) + fibonacci(k-2)
```

What is the time complexity of computing `fibonacci(k)` and reducing it mod  $p$ ?

- (h) If you modify the recursive algorithm in (g) to return  $\text{GF}(p)$  (1) in the base case, it will work in  $\mathbb{F}_p$  rather than  $\mathbb{Z}$ . What is its complexity then?

## Problem 2. Cornacchia's algorithm (20 points)

Cornacchia's algorithm computes primitive solutions  $(x, y)$  to the Diophantine equation

$$x^2 + dy^2 = m, \quad (2)$$

where  $d$  and  $m$  are positive integers. A *primitive* solution has  $x$  and  $y$  relatively prime. Typically  $m = p$  or  $m = 4p$ , where  $p$  is a prime, but the algorithm works for any  $m$ , provided we are given a square root of  $-d \pmod{m}$ .

The algorithm uses a partial Euclidean algorithm that terminates as soon as the sequence of remainders  $r_i$  drops below the square root of  $r_0 = m$ .

1. Let  $r_0 = m$  and  $r_1 = r$ , where  $r^2 \equiv -d \pmod{m}$  and  $0 \leq r \leq m/2$ .
2. Compute  $r_{i+2} = r_i \pmod{r_{i+1}}$  until  $r_k^2 < m$  is reached.
3. If  $(m - r_k^2)/d$  is the square of an integer  $s$ , return the solution  $(r_k, s)$ . Otherwise, return `null`.

It is clear that if the algorithm returns  $(r_k, s)$ , then it is a solution to (2). It is not so clear that the algorithm always finds a primitive solution if one exists, but this is true; see [1] for a short elementary proof.<sup>2</sup> If  $m$  is square-free (as when  $m$  is prime), every solution to (2) is primitive, but this is not true in general (this is relevant to part (e)).

In this problem let  $N := n \cdot 10^{100}$ , where  $n$  is the last 4 digits of your student ID.

- (a) Implement this algorithm in Sage. Use `mod(-d, m).is_square()` to test if  $-d$  has a square root mod  $m$ , and use `int(mod(-d, m).sqrt())` to get a square root.
- (b) You may recall Fermat's "Christmas theorem", which states that an odd prime  $p$  is the sum of two squares if and only if  $p \equiv 1 \pmod{4}$ . You may also recall that  $-1$  is a square modulo an odd prime  $p$  if and only if  $p \equiv 1 \pmod{4}$ .

Let  $n$  be the integer corresponding to the last 4 digits of your student ID. For the least prime  $p > N$  congruent to  $1 \pmod{4}$ , write  $p$  as the sum of two squares.

- (c) Fermat also proved that a prime  $p$  can be written in the form  $p = x^2 + 3y^2$  if and only if  $p \equiv 1 \pmod{3}$ , which is equivalent to the condition that  $-3$  is a square mod  $p$ . For the least prime  $p > N$  congruent to  $1 \pmod{3}$ , write  $p$  in the form  $p = x^2 + 3y^2$ .
- (d) Show that this does not work for  $d = 5$  by finding a prime  $p$  for which  $-5$  is a square modulo  $p$  but  $p$  cannot be written in the form  $x^2 + 5y^2$ . Empirically determine a stronger congruence condition on  $p$  that guarantees not only that  $-5$  is a square mod  $p$ , but also that  $p$  can be written in the form  $x^2 + 5y^2$ . Then find the least prime  $p > N$  that satisfies your condition and write  $p$  in the form  $p = x^2 + 5y^2$ .

<sup>2</sup>There is an obvious typo in step 3 of the algorithm given in [1] which is corrected above.

- (e) Let  $E$  be the elliptic curve  $y^2 = x^3 - 35x - 98$ . As you found in Problem Set 1, the integer  $a_p = p + 1 - \#E(\mathbb{F}_p)$  is zero precisely when  $-7$  is not a square modulo  $p$ . When  $-7$  is a square modulo  $p$ , the integer  $a_p$  satisfies the equation  $4p = a_p^2 + 7y^2$ , for some  $y \in \mathbb{Z}$ . Prove that this equation has no primitive solutions for  $p > 2$ , and in this case it has a solution if and only if the equation  $p = u^2 + 7v^2$  has a solution.
- Use your algorithm to find a solution to  $p = u^2 + 7v^2$  for the least prime  $p > N$  for which  $-7$  is a square modulo  $p$ , and use this to deduce the absolute value of  $a_p$ . Determine the sign of  $a_p$ , by finding a random point  $P \in E(\mathbb{F}_p)$  for which only one of  $(p + 1 - a_p)P$  and  $(p + 1 + a_p)P$  is zero.

### Problem 3. Computing $r$ th roots in cyclic groups (40 points)

In Lecture 4 we saw how to compute  $r$ th roots in a finite field  $\mathbb{F}_q$  using a probabilistic root-finding algorithm. In this problem you will implement an entirely different approach for computing  $r$ th roots that works in any finite cyclic group  $G$ , including  $G = \mathbb{F}_q^\times$ . In addition to being more general, this method is typically faster than using probabilistic root-finding to compute  $r$ th roots in  $\mathbb{F}_q^\times$  (but this depends on the values of  $r$  and  $q$ ).

We assume without loss of generality that  $r$  is prime (to compute  $n$ th roots, successively compute  $r$ th roots for the primes  $r$  dividing  $n$ , with multiplicity). To simplify notation we write  $G$  additively, so an  $r$ th root of  $\gamma \in G$  is an element  $\rho \in G$  for which  $r\rho = \gamma$ . Let  $|\gamma|$  denote the *order* of  $\gamma$ , the least positive integer  $m$  for which  $m\gamma = 0$ .

**Prove the following statements:**

- (a) For all  $\gamma \in G$  and  $n \in \mathbb{Z}$  we have  $|n\gamma| = |\gamma|/\gcd(n, |\gamma|)$ .
- (b) If  $r$  does not divide  $|G|$ , then there is an integer  $s$  such that for all  $\gamma \in G$  the element  $\rho = s\gamma$  is the unique  $r$ th root of  $\gamma$ .
- (c) If  $r$  does divide  $|G|$ , then the number of  $r$ th roots of each  $\gamma \in G$  is either 0 or  $r$ . In the latter case, the  $r$ th roots of  $\gamma$  do not necessarily lie in  $\langle \gamma \rangle$  (give an example).
- (d) Suppose  $r$  divides  $|G|$ . Let  $|G| = ar^k$ , where  $r \nmid a$ . Let  $\delta \in G$  be an element of order  $r^k$ , let  $\gamma$  be any element of  $G$ , and  $\alpha = a\gamma$  and  $\beta = r^k\gamma$ . The following hold:
- (i)  $\alpha = x\delta$  for some integer  $x \in [1, r^k]$ .
  - (ii) If  $r$  does not divide  $x$  then there is no  $\rho \in G$  for which  $r\rho = \gamma$ .
  - (iii) If  $r$  divides  $x$ , and  $s$  and  $t$  are integers satisfying  $sa + tr^{k+1} = 1$ , then the element  $\rho = s(x/r)\delta + t\beta$  satisfies  $r\rho = \gamma$ .

The element  $\delta$  is a generator for the  $r$ -Sylow subgroup of  $G$ . If  $G = \langle \sigma \rangle$ , we can use  $\delta = a\sigma$ . The integer  $x$  is the *discrete logarithm* of  $\alpha$  with respect to  $\delta$ .

**Implement the following algorithm** for computing an  $r$ th root of  $\gamma$  in a cyclic group  $G$  of order  $ar^k$ , where  $r$  is a prime that does not divide  $a$ , given  $\delta \in G$  of order  $r^k$ :

1. If  $k = 0$  then compute  $s = 1/r \pmod a$  and return  $\rho = s\gamma$ .
2. Compute  $\alpha = a\gamma$  and  $\beta = r^k\gamma$ .
3. Compute the discrete logarithm  $x$  of  $\alpha$  with respect to  $\delta$  by brute force: check whether  $\alpha = x\delta$  for each  $x$  from 1 to  $r^k$  (this holds for some  $x$ , by part (a) of 4).<sup>3</sup>

<sup>3</sup>We will learn much better ways to compute this discrete logarithm later in the course. For the moment, assume  $r^k$  is small (for finite fields  $\mathbb{F}_q$ , this is usually the case, even when  $q$  is very large).

4. If  $r$  does not divide  $x$  then return `null`.
5. Compute  $s$  and  $t$  such that  $sa + tr^{k+1} = 1$  using the extended Euclidean algorithm.
6. Return  $\rho = s(x/r)\delta + t\beta$ .

The return value `null` is used to indicate that  $\gamma$  does not have any  $r$ th roots in  $G$ . To compute  $s = 1/r \pmod a$  in Sage, use: `s=1/mod(r,a)`. To compute  $s$  and  $t$  such that  $sa + tr^{k+1} = 1$ , use: `d,s,t=xgcd(a,r**(k+1))` (here  $d = \gcd(a, r^{k+1})$  is 1).

The Python language used by Sage is untyped, so your algorithm can be used to compute  $r$ th roots in any cyclic group that Sage knows how to represent; it will automatically perform operations in whatever group the inputs  $\delta$  and  $\gamma$  happen to lie in. To test your algorithm, you may find it useful to work in the additive group of the ring  $\mathbb{Z}/n\mathbb{Z}$ , where  $n = ar^k$ , which you can create in Sage using `Zn=Integers(n)`. You can then use `delta=Zn(a)` to create an element of  $\mathbb{Z}/n\mathbb{Z}$  with additive order  $r^k$ .

Let  $E$  be the elliptic curve  $y^2 = x^3 + 31415926x + 21782818$  over  $\mathbb{F}_p$  with  $p = 2^{255} - 19$ . The group  $E(\mathbb{F}_p)$  is cyclic, of order  $n = 2 \cdot 5 \cdot 11 \cdot m$ , where

$$m = 526327678351437251925322659130399581153890530360799936898619298682575411573,$$

and the point  $P = (x_0 : y_0 : 1)$ , where  $x_0 = 1$  and

$$y_0 = 13134814009004874435178595602484727393250015841439437156743965100178108553104,$$

is a generator for  $E(\mathbb{F}_p)$ . Thus for  $r = 2, 5, 11$  you can use  $\delta = (n/r)P$  as a generator of the  $r$ -Sylow subgroup (which in each case has order  $r$ ).

Let  $c$  be the least prime greater than the integer formed by the last four digits of your student ID. Let  $Q = (x_1 : y_1 : 1)$ , where

$$\begin{aligned} x_1 &= 6044428498752310014675982582859197165940793807815638882398853158746179998524, \\ y_1 &= 15213333001572637024476234182397935874270965424629854864096629656428211884435 \end{aligned}$$

- (e) Use your algorithm to find an  $r$ th root  $R$  of  $\gamma = cQ$ , for  $r = 2, 5, 11$ . Note that you can easily check your result by testing whether `r*R==c*Q` holds using Sage (please be sure to do this). In your answer you only need to list the point  $R$  for each value of  $r$ , you don't need to include your code. Be sure to format your answer so that the coordinates of  $R$  all fit on the page.

#### Problem 4. Exponentiating with addition chains (40 points)

An *addition chain* for a positive integer  $n$  is an increasing sequence of integers  $(c_0, \dots, c_m)$  with  $c_0 = 1$  and  $c_m = n$  such that each entry other than  $c_0$  is the sum of two (not necessarily distinct) preceding entries. The *length* of an addition chain is the index  $m$  of the last entry. When computing  $a^n$  with a generic algorithm, the exponents of the powers of  $a^k$  computed by the algorithm define an addition chain whose length is the number of multiplications performed. For example, using left-to-right binary exponentiation to compute  $a^{47}$  yields the addition chain  $(1, 2, 4, 5, 10, 11, 22, 23, 46, 47)$ , and right-to-left binary exponentiation yields the addition chain  $(1, 2, 3, 4, 7, 8, 15, 16, 32, 47)$ , both of which have length 9.

- (a) For  $n = 715$ , determine the addition chains given by: (i) left-to-right binary, (ii) right-to-left binary, (iii) fixed-window, and (iv) sliding-window exponentiation, using a window of size 2 for (iii) and (iv).

- (b) Find an addition chain for  $n = 715$  that is shorter than any you found in part (a).
- (c) Repeat part (a) for the integer  $N$  obtained by adding 990,000 to the last 4 digits of your student ID, using a window of size 3 for cases fixed and sliding window cases.
- (d) Find the shortest addition chain for  $N$  that you can. There is a good chance you can do better than any of the chains you found in part (c).

In groups where inversions are cheap (such as the group of points on an elliptic curve), it is advantageous to use *signed* binary representations of exponents, where we write the exponent  $n$  in the form  $n = \sum n_i 2^i$  with  $n_i \in \{-1, 0, 1\}$ . Such a representation is generally not unique, but there is a unique signed representation with the property that no pair of adjacent digits are both nonzero. This is known as *non-adjacent form* (NAF). The NAF representation of 47, for example, is  $10\bar{1}000\bar{1}$ , where  $\bar{1}$  denotes  $-1$ .

To construct the NAF representation one begins by writing  $n$  in binary with a leading 0, and then successively replaces the least significant block of the form  $01 \cdots 1$  with  $10 \cdots 0\bar{1}$  until there are no adjacent nonzero digits. For example, the computation for 47 proceeds as  $0101111, 11000\bar{1}, 10\bar{1}000\bar{1}$ , which reduces the number of nonzero digits from 5 to 3. Even though the length is increased by 1, the total cost of exponentiation may be reduced.

An *addition-subtraction* chain extends the definition of an addition chain by allowing  $c_k = c_i \pm c_j$ . Exponentiation using the NAF representation defines an addition-subtraction chain. For example, using left-to-right binary exponentiation, the NAF representation of 47 yields the chain  $(1, 2, 4, 3, 6, 12, 24, 48, 47)$ , which is shorter than the addition chain  $(1, 2, 4, 5, 10, 11, 22, 23, 46, 47)$  given by standard left-to-right binary exponentiation.

- (e) Compute addition-subtraction chains for  $n = 715$  and the integer  $N$  defined in part (c) using left-to-right binary exponentiation with the NAF representation.
- (f) Find the shortest addition-subtraction chains for  $n$  and  $N$  that you can.

### Problem 5. Root-finding over $\mathbb{Z}$ (80 points)

In this problem you will develop an algorithm to find integer roots of polynomials in  $\mathbb{Z}[x]$  using a  $p$ -adic version of Newton's method (also known as Hensel lifting). As an application, this gives us an efficient way to factor perfect powers (a special case that we will need to handle when we come to the elliptic curve factorization method), and it will be used as a black box in Problem 2 to find integer roots of division polynomials.

In the questions below,  $p$  can be any integer greater than 1, but you may assume it is a prime power if you wish.

- (a) Let  $x_0 \in \mathbb{Z}$  and  $f \in \mathbb{Z}[x]$ . Prove that the following equivalence holds in  $\mathbb{Z}[x]$ :

$$f(x) \equiv f(x_0) + f'(x_0)(x - x_0) \pmod{(x - x_0)^2}.$$

- (b) Let  $x_0, z_0 \in \mathbb{Z}$  and  $f \in \mathbb{Z}[x]$  satisfy  $f(x_0) \equiv 0 \pmod{p}$  and  $f'(x_0)z_0 \equiv 1 \pmod{p}$ . Let

$$\begin{aligned} x_1 &\equiv x_0 - f(x_0)z_0 \pmod{p^2}, \\ z_1 &\equiv 2z_0 - f'(x_1)z_0^2 \pmod{p^2}. \end{aligned}$$

Prove that that the following three equivalences hold:

$$\begin{aligned} x_1 &\equiv x_0 \pmod{p}, & \text{(i)} \\ f(x_1) &\equiv 0 \pmod{p^2}, & \text{(ii)} \\ f'(x_1)z_1 &\equiv 1 \pmod{p^2}. & \text{(iii)} \end{aligned}$$

Show that (i) and (ii) characterize  $x_1 \pmod{p^2}$  uniquely by proving that if  $x_2 \in \mathbb{Z}$  also satisfies  $x_2 \equiv x_0 \pmod{p}$  and  $f(x_2) \equiv 0 \pmod{p^2}$ , then  $x_1 \equiv x_2 \pmod{p^2}$ .

Iteratively applying (b) yields an algorithm that, given an integer  $k$  and  $x_0, z_0$ , and  $f$  satisfying the hypothesis (a), outputs an integer  $x_k$  that satisfies  $f(x_k) \equiv 0 \pmod{p^{2^k}}$ .

- (c) Prove that if  $f$  has an integer root  $r$  for which  $f'(r)$  is invertible modulo  $p$ , then given  $x_0 \equiv r \pmod{p}$ ,  $z_0 \equiv 1/f'(x_0) \pmod{p}$ , and  $k$  such that  $|r| < p^{2^k}/2$ , this algorithm outputs  $x_k$  such that  $r$  is the unique integer  $r \equiv x_k \pmod{p^{2^k}}$  satisfying  $|r| < p^{2^k}/2$ .

To apply the result in (c), we need to know a suitable starting value (or values) for  $x_0$ . For the two applications we have in mind, this is will be straightforward, so let us proceed on the assumption that we are given a suitable  $x_0$  with  $f'(x_0)$  invertible modulo  $p$ .

Let  $B$  be the maximum of the absolute values of the coefficients of  $f$ , and let  $B_0$  be an upper bound on the absolute value of its largest integer root. It suffices to choose the least  $k$  such that  $p^{2^k} > 2B_0$ , and since any integer root of  $f$  must divide its constant coefficient, we can assume that  $B_0 \leq B$ . We can also assume  $p < 2B_0$ , since otherwise the problem is trivial ( $k = 0$  and  $x_k = x_0$ ).

- (d) Prove that with this choice of  $k$  the algorithm can be implemented to run in time  $O(d M(\log B))$ , where  $d$  is the degree of  $f$  (be careful here, the most obvious implementation will not achieve this time bound). Prove that if  $f$  has  $O(1)$  terms, then the algorithm can be implemented to run in time  $O(M(\log B) + M(\log B_0) \log d)$ .
- (e) Using the primes  $p = 2$  and  $p = 3$ , describe an efficient algorithm that, given an integer  $N$  relatively prime to 6, either outputs an integer  $a$  and a prime  $q$  such that  $a^q = N$ , or proves that  $N$  is not a perfect power. Prove that your algorithm runs in quasi-quadratic time (meaning  $O(n^2(\log n)^c)$  for some constant  $c$ , where  $n = \log N$ ).
- (f) Implement your algorithm and report the result and running time on the each of the following inputs:  $2^{1000} + 297$ ,  $5^{503}$ ,  $(2^{500} + 55)^2$ ,  $(2^{333} + 285)^3$ ,  $(2^{32} + 15)^{31}$ ,  $100! + 1$ . To time your code in sage, use the `timeit` function (e.g. `timeit("1+1")`).
- (g) Prove that the algorithm you gave in (e) can be implemented to run in sub-quadratic time (meaning  $o(n^2)$  where  $n = \log N$ ). You may need to modify your algorithm in order to achieve this.<sup>4</sup>

## Problem 6. Survey

Complete the following survey by rating each of the problems you attempted on a scale of 1 to 10 according to how interesting you found the problem (1 = “mind-numbing,” 10 = “mind-blowing”), and how difficult you found it (1 = “trivial,” 10 = “brutal”). Also estimate the amount of time you spent on each problem to the nearest half hour.

<sup>4</sup>In fact, this problem can be solved in quasi-linear time [2].

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Problem 5			

Also, please rate each of the following lectures that you attended, according to the quality of the material (1=“useless”, 10=“fascinating”), the presentation (1=“epic fail”, 10=“perfection”), the pace (1=“way too slow”, 10=“way to fast”), and the novelty of the material (1=“old hat”, 10=“all new”).

Date	Lecture Topic	Material	Presentation	Pace	Novelty
2/21	Finite field arithmetic / finite fields				
2/22	Isogenies				

Feel free to record any additional comments you have on the problem sets or lectures.

## References

- [1] Julius Magalona Basilla, *On the solution of  $x^2 + dy^2 = m$* , Proc. Japan Acad. Ser. A Math Sci. **80** (2004), 40–41.
- [2] D. J. Bernstein, *Detecting perfect powers in essentially linear time*, Math. Comp. **67** (1998), 1252–1283.