18.745 Introduction to Lie Algebras	October 5, 2010

Lecture 8 — Cartan Subalgebra

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Definition 8.1. Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. Then $N_{\mathfrak{g}}(\mathfrak{h}) = \{a \in \mathfrak{g} | [a, \mathfrak{h}] \subset \mathfrak{h}\}$ is a subalgebra of $\mathfrak{g}$, called the normalizer of $\mathfrak{h}$.

The fact that $N_{\mathfrak{g}}(\mathfrak{h})$ is a subalgebra follows directly from the Jacobi identity. Also note that $\mathfrak{h} \subset N_{\mathfrak{g}}(\mathfrak{h})$ and the normalizer of $\mathfrak{h}$ is the maximal subalgebra containing $\mathfrak{h}$ as an ideal.

Lemma 8.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra such that $\mathfrak{h} \neq \mathfrak{g}$. Then, $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$.*

Proof. Consider the central series: $\mathfrak{g} = \mathfrak{g}^1 \subset \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \subset g^3 \subset ... \subset g^n = 0$
Note that the last equality is true for some $n \in \mathbb{N}$ because $\mathfrak{g}$ is a nilpotent Lie algebra . Take j to be the maximal possible positive integer such that: $\mathfrak{g}^j \not\subseteq \mathfrak{h}$. Clearly we have that $1 < j < n$; but then $[\mathfrak{g}^j, \mathfrak{h}] \subset \mathfrak{g}^{j+1} \subset \mathfrak{h}$ by the choice made on j. Hence $\mathfrak{g}^j \subset N_{\mathfrak{g}}(\mathfrak{h})$, which is not a subspace of $\mathfrak{h}$. Thus, we can conclude that $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$ □

Definition 8.2. A Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$, satisfying the following two conditions:
i) $\mathfrak{h}$ is a nilpotent Lie algebra
ii) $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$

Corollary 8.2. *Any Cartan subalgebra of $\mathfrak{g}$ is a maximal nilpotent subalgebra*

Proof. This follows directly from Lemma 1 and the definition of Cartan subalgebras.

□

Exercise 8.1. Let $\mathfrak{g} = gl_n(\mathbb{F})$ with $char(\mathbb{F}) \neq 2$. Let $\mathfrak{h} = \{\mathfrak{n}_n + \mathbb{F}I_n\}$, where $\mathfrak{n}_n$ is the subalgebra of strictly upper triangular matrices. Then this is a maximal nilpotent subalgebra but not a Cartan subalgebra.

Solution. First, we show that $\mathfrak{n}_n + \mathbb{F}I_n$ is not a Cartan subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ is not Cartan since, as was shown earlier, $[b_n, \mathfrak{n}_n] \subset \mathfrak{n}_n$ and thus $b_n \subset N_{\mathfrak{g}}(\mathfrak{n}_n) \subset N_{\mathfrak{g}}(\mathfrak{n}_n + \mathbb{F}I_n)$.
Now we show that $\mathfrak{n}_n + \mathbb{F}I_n$ is a maximal nilpotent subalgebra. Note that since I_n commutes with everything, then $\mathfrak{h} = \mathfrak{n}_n \bigoplus \mathbb{F}I_n$. Hence, it is nilpotent. Now, it suffices to show that $\mathfrak{h}$ is maximal in gl_n. Suppose there exists some nilpotent subalgebra $\mathfrak{n}_n + \mathbb{F}I_n \subsetneq \mathfrak{h}'$. First note that in fact we have $b_n = N_{gl_n}(\mathfrak{n}_n + \mathbb{F}I_n)$, since if $b = \sum_{i,j} c_{ij}E_{ij} \in N_{\mathfrak{g}}(\mathfrak{n}_n + \mathbb{F}I_n)$ with $c_{i'j'} \neq 0$ for $i' > j'$, then $E_{j'i'} \in \mathfrak{n}_n \subset \mathfrak{n}_n + \mathbb{F}I_n$ and $[b, E_{j'i'}] = \sum_i c_{ij'}E_{ii'} - \sum_j c_{i'j}E_{j'j}$. Note that the $(i', i')^{th}$ and the $(j', j')^{th}$ entries are $c_{i'j'}$ and $-c_{i'j'}$ respectively; both of which are nonzero. Thus, $[b, E_{j'i'}]\mathfrak{n}_n + \mathbb{F}I_n$, unless $char(\mathbb{F}) = 2$. Now, by the Lemma above, we must have that $\mathfrak{n}_n + \mathbb{F}I_n \subsetneq \mathfrak{h} \cap N_{gl_n}(\mathfrak{n}_n + \mathbb{F}I_n)$. Thus, $\mathfrak{h}'$ contains some element of $b_n \backslash \mathfrak{n}_n + \mathbb{F}I_n$; but all elements of $b_n \backslash \mathfrak{n}_n + \mathbb{F}I_n$ have at least two distinct eigenvalues, and thus are not ad-nilpotent. Thus we find a contradiction to Engel's theorem and our assumption must be wrong. We find there is no proper set containing $\mathfrak{n}_n + \mathbb{F}I_n$; which implies that $\mathfrak{h}$ is a maximal nilpotent subalgebra.

Proposition 8.3. *Let $\mathfrak{g} \subset gl_n(\mathbb{F})$ be a subalgebra containing a diagonal matrix $a = diag(a_1, ..., a_n)$ with distinct a_i, and let $\mathfrak{h}$ be the subspace of all diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra.*

Proof. We have to prove that $\mathfrak{h}$ satisfies the two conditions necessary to be a Cartan subalgebra.
i) We know that $\mathfrak{h}$ is abelian; and thus, it is a nilpotent Lie algebra.
ii)Let $b = \sum_{i,j=1}^{n} b_{ij} e_{ij} \in \mathfrak{g}$ such that $[b, \mathfrak{h}] \subset \mathfrak{h}$ (i.e. $b \in N_{\mathfrak{g}}(\mathfrak{h})$). In particular, we have that $[a, b] \in \mathfrak{h}$ for all $a \in \mathfrak{h}$. But $[a, b] = [\sum_k a_k e_{kk}, \sum_{i,j} e_{ij} e_{ij}] = \sum_{ij}(a_i - a_j) b_{ij} e_{ij}$, which will be non-diagonal only if $b_{ij} \neq 0$ for some $i \neq j$. Thus, we can conclude that $b \in \mathfrak{h}$, and therefore $N_g(\mathfrak{h}) = \mathfrak{h}$. □

Remark $\mathfrak{g}$ is a Cartan subalgebra in itself if and only if $\mathfrak{g}$ is a nilpotent Lie algebra.

Theorem 8.4. *(E.Cartan) Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$. Let $a \in \mathfrak{g}$ be a regular element (which exists since $\mathbb{F}$ is infinite), and let $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_\lambda^a$ be the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ada. Then $\mathfrak{h} = \mathfrak{g}_0^a$ is a Cartan subalgebra.*

Proof. The proof for this theorem uses the fact that Zariski Topology is highly non-Hausdorff, namely any two non-empty Zariski open sets have a non-empty intersection. We will also recall the fact that $[\mathfrak{g}_\lambda^a, \mathfrak{g}_\mu^a] \subset \mathfrak{g}_{\lambda+\mu}^a$ and in particular, if $\lambda = 0$ then $[\mathfrak{h}, \mathfrak{g}_\mu^a] \subset \mathfrak{g}_\mu^a$.
Let $V = \bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda^a$. Then $\mathfrak{g} = \mathfrak{h} \bigoplus V$ and $[\mathfrak{h}, V] \subset V$.
Consider the following two subsets of $\mathfrak{h}$:
$U = \{h \in \mathfrak{h}$ such that $adh|_{\mathfrak{h}}$ is not a nilpotent operator$\}$
$R = \{h \in \mathfrak{h}$ such that $adh|_V$ is a non-singular operator $\}$
Both U and R are Zariski open subsets of $\mathfrak{h}$. Next we note that $a \in R$ since all zero eigenvalues of ada lie in $\mathfrak{h}$, hence R is non-empty.
Now, we shall prove by contradiction that $\mathfrak{h}$ is a nilpotent subalgebra. Suppose the contrary is true. Then, by Engel's Theorem there exists $h \in \mathfrak{h}$ such that $adh|_{\mathfrak{h}}$ is not nilpotent. But in this case $h \in U$; and hence, $U \neq \emptyset$. Therefore $U \bigcap R \neq \emptyset$. We now take $b \in U \bigcap R$. Then $adh|_{\mathfrak{h}}$ is not nilpotent and $adb|_V$ is invertible. Hence $\mathfrak{g}_0^b \subsetneq \mathfrak{h}$, which contradicts the fact that a is a regular element. Thus, this contradicts the assumption made; and we find that $\mathfrak{h}$ is a nilpotent Lie algebra.
Finally, we need to proof that $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Now, if $b \in N_{\mathfrak{g}}(\mathfrak{h})$, so that $[b, \mathfrak{h}] \subset \mathfrak{h}$, then we have that, in particular, $[b, a] \in \mathfrak{h}$. But since $a \in \mathfrak{h}$ and $\mathfrak{h}$ is a nilpotent Lie algera, then $ada|_{\mathfrak{h}}$ is a nilpotent operator. In particular, $0 = (ada)^N((ada)b) = (ada)^{N+1}(b)$. Hence $b \in \mathfrak{g}_0^a = \mathfrak{h}$, which completes the proof of the theorem. □

Remark The dimension of the Cartan subalgebra constructed in Cartan's Theorem, Theorem 4, equals the rank of $\mathfrak{g}$.

Proposition 8.5. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Consider the generalized weight space decomposition (called root space decomposition) with respect to $\mathfrak{h}$: $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$. Then $\mathfrak{g}_0 = \mathfrak{h}$.*

Proof. Since by Engel's Theorem $adh|_{\mathfrak{h}}$ is nilpotent for all $h \in \mathfrak{h}$, it follows that $\mathfrak{h} \subseteq \mathfrak{g}_0$. But by definition of $\mathfrak{g}$, for all elements $h \in \mathfrak{h}$, $adh|_{\mathfrak{g}_0}$ is a nilpotent operator. Hence $adh|_{\mathfrak{g}_0/\mathfrak{h}}$ is a nilpotent operator for all $h \in \mathfrak{h}$. Therefore, by Engle's Theorem, there exists a non-zero $barb \in \mathfrak{g}_0/\mathfrak{h}$ which is annihilated by all $adh|_{\mathfrak{g}_0/\mathfrak{h}}$. Taking a pre-image $b \in \mathfrak{g}$ of $\bar{b}$, this means that $[b, \mathfrak{h}] \subset \mathfrak{h}$ (i.e. $\mathfrak{h} \neq N_{\mathfrak{g}}(\mathfrak{h})$), which contradicts the fact that $\mathfrak{h}$ is a Cartan subalgebra. □

Remark $\mathfrak{g} = \mathfrak{h} \bigoplus (\bigoplus_{\lambda \in \mathfrak{h}^*, \lambda \neq 0} \mathfrak{g}_\lambda)$ by the latter Proposition.

Next we will apply the last couple of theorems, lemmas and propositions in order to classify all 3-dimensional Lie algebras $\mathfrak{g}$ over an algebraically closed field $\mathbb{F}$ of characteristic zero.
We know that the $rank(\mathfrak{g}) = 3, 2$ or 1; and $rank(\mathfrak{g}) = 3$ if and only if $\mathfrak{g}$ is nilpotent.
$Rank(\mathfrak{g}) = 3$:
We know by exercise 6.1, that any 3-dimensional nilpotent Lie algebra is either abelian or H_3. So any 3-dimensional Lie algebra of $rank(\mathfrak{g}) = 3$ is either abelian or H_3.
$Rank(\mathfrak{g}) = 2$:
In this case $dim(\mathfrak{h}) = 2$. Since $\mathfrak{h}$ must be a nilpotent Lie algebra, we can conclude that $\mathfrak{h}$ is abelian (otherwise $\mathfrak{h}$ would be a 2-dimensional solvable Lie algebra, which is not nilpotent). Hence the root space decomposition is $\mathfrak{g} = \mathfrak{h} \bigoplus \mathbb{F}b$, where $[\mathfrak{h}, b] \subset \mathbb{F}b$. Since $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$ then $[\mathfrak{h}, b] \neq 0$. Hence there exist $a \in \mathfrak{h}$ such that $[a, b] = b$. Also since $\mathfrak{h}$ is 2-dimensional, then there exists $c \in \mathfrak{h}$ such that $[c, b] = 0$. We also know that $[a, c] = 0$. Thus, we can conclude that the only 3-dimensional Lie algebra of rank 2 is a direct sum of a 2-dimensional non-abelian algebra and one dimensional central subalgebra: $\mathfrak{g} = (\mathbb{F}a + \mathbb{F}b) \bigoplus \mathbb{F}c$.
$Rank(\mathfrak{g}) = 1$:

Exercise 8.2. Any 3-dimensional Lie algebra $\mathfrak{g}$ of rank 1 is isomorphic to one of the following Lie algebras with basis h, a, b:
i) $[h, a] = a$, $[h, b] = a + b$, $[a, b] = 0$;
ii) $[h, a] = a$, $[h, b] = \lambda b$, where $\lambda \in \mathbb{F}/\{0\}$ $[a, b] = 0$;
iii) $[h, a] = a$, $[h, b] = -b$, $[a, b] = h$;

Solution. Let $\mathfrak{h} = \mathbb{F}h$,where $h \in \mathfrak{g}$, be a Cartan subalgebra; we have: $\mathfrak{g} = \mathfrak{h} \bigoplus V$, where $[h, V] \subset V$, $dim(V) = 2$, and adh is non-singular on V. We may assume one of the eigenvalues of $adh|_V$ is 1, if we scale h accordingly.

First, suppose $adh|_V$ is not semisimple. Thus, $adh|_V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$in some basis $\{a, b\}$. Thus, $[h, a] = a$ and $[h, b] = a + b$. Also, by using Jacobian Identity, we have $[h, [a, b]] = [[h, a], b] + [a, [h, b]] = [a, b] + [a, a + b] = 2[a, b]$. But we know that the value 2 cannot be an eigenvalue of adh, thus $[a, b] = 0$. Thus, this Lie alebra corresponds to (i).
Now, assume $adh|_V$ is semisimple. Thus, we have that $adh|_V = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$in some basis $\{a, b\}$. We then have that $[h, a] = a$ and $[h, b] = \lambda b$. By following the same procedure as above, we note that $[h, [a, b]] = [[h, a], b] + [a, [h, b]] = [a, b] + [a, \lambda b] = (1 + \lambda)[a, b]$. Thus, we must have that either $[a, b] = 0$, which corresponds to (ii), or $(1 + \lambda)$ is an eigenvalue of $adh|_V$, case (iii). Note that for the case of $[a, b] = 0$, then λ is arbitrary and uniquely defined by $\mathfrak{g}$ up to inverting it by swapping a and b and scaling h accordingly. In the case were $(1 + \lambda)$ is an eigenvalue of $adh|_V$, we must have that $1 + \lambda = 0$ and that $[a, b]$ be a multiple of h. Thus $\lambda = -1$ and, scaling a accordingly, we may assume that $[a, b] = h$. Thus, we get option (iii) with the latter case.

Exercise 8.3. Show that all Lie algebras in exercise 8.2 are non-isomorphic . Those from (i) and (ii) are solvable, and the one from (iii) is isomorphic to $sl_2(\mathbb{F})$, which is not solvable.

Solution. We first show that the Lie algebra from (iii) is isomorphic to $sl_2(\mathbb{F})$. The standard basis for $sl_2(\mathbb{F})$ consist of { h'= $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $a' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$}.Thus, we have

$[h, a'] = 2a', [h, b'] = -2b'$ and $[a', b'] = h$. Now if we scale and set $h = \frac{h'}{2}$, $a = \frac{a'}{\sqrt{2}}$ and $b = \frac{b'}{\sqrt{2}}$; then we get the Lie algebra of case (iii). Thus, Lie algebra (iii) is isomorphic to $sl_2(\mathbb{F})$, which is not solvable. In contrast, it is clear that the algebras (i) and (ii) are solvable by construction.

By conjugacy of Cartan subalgebras, the isomorphism class of $\mathfrak{g}$ needs to be independent of the choice of the Cartan subalgebra. It follows that the algebras of the three types are non-isomorphic to each other. It also follows that the algebras of type (ii), corresponding to parameters λ and λ' are isomorphic if and only if $\lambda\lambda' = 1$