18.745 Introduction to Lie Algebras September 28, 2010

Lecture 6 — Generalized Eigenspaces & Generalized Weight Spaces

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Definition 6.1. Let A be a linear operator on a vector space V over field $\mathbb{F}$ and let $\lambda \in \mathbb{F}$, then the subspace

$$V_\lambda = \{v \mid (A - \lambda I)^N v = 0 \text{ for some positive integer N}\}$$

is called a generalized eigenspace of A with eigenvalue λ. Note that the eigenspace of A with eigenvalue λ is a subspace of V_λ.

Example 6.1. A is a nilpotent operator if and only if $V = V_0$.

Proposition 6.1. *Let A be a linear operator on a finite dimensional vector space V over an algebraically closed field $\mathbb{F}$, and let $\lambda_1, ..., \lambda_s$ be all eigenvalues of A, $n_1, n_2, ..., n_s$ be their multiplicities. Then one has the generalized eigenspace decomposition:*

$$V = \bigoplus_{i=1}^{s} V_{\lambda_i} \text{ where } \dim V_{\lambda_i} = n_i$$

Proof. By the Jordan normal form of A in some basis $e_1, e_2, ...e_n$. Its matrix is of the following form:

$$A = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_n} \end{pmatrix},$$

where J_{λ_i} is an $n_i \times n_i$ matrix with λ_i on the diagonal, 0 or 1 in each entry just above the diagonal, and 0 everywhere else.

Let $V_{\lambda_1} = \text{span}\{e_1, e_2, ..., e_{n_1}\}, V_{\lambda_2} = \text{span}\{e_{n_1+1}, ..., e_{n_1+n_2}\}, ...$, so that J_{λ_i} acts on V_{λ_i}. i.e. V_{λ_i} are A-invariant and $A|_{V_{\lambda_i}} = \lambda_i I_{n_i} + N_i$, N_i nilpotent. □

From the above discussion, we obtain the following decomposition of the operator A, called the classical Jordan decomposition

$$A = A_s + A_n$$

where A_s is the operator which in the basis above is the diagonal part of A, and A_n is the rest ($A_n = A - A_s$). It has the following 3 properties

(i) A_s is a diagonalizable operator (usually called semisimple)

(ii) A_n is a nilpotent operator

(iii) $A_s A_n = A_n A_s$.

(iii) holds since $V = \bigoplus_{i=1}^{s} V_{\lambda_i}$, $AV_{\lambda_i} \in V_{\lambda_i}$, and $A_s|_{V_{\lambda_i}} = \lambda_i I$. Hence $A_s A_n = A_n A_s$.

Definition 6.2. A decomposition of an operator A of the form $A = A_s + A_n$, for which these three properities hold is called a Jordan decomposition of A. We have established its existence, provided that $\dim V < +\infty, \mathbb{F} = \bar{\mathbb{F}}$

Proposition 6.2. *Jordan decomposition is unique under the same assumptions*

Lemma 6.3. *Let A and B be commuting operators on V; i.e., $AB = BA$. Then*

(a) All generalized eigenspaces of A are B-invariant

(b) if $A = A_s + A_n$ is the classical Jordan decomposition, then B commutes with both A_s and A_n.

Proof. (a) is immediate from definition of generalized eigenspace. (b) follows from (a) since each V_{λ_i} is B-invariant, $A_s|_{V_{\lambda_i}} = \lambda_i I_{n_i}$, therefore A and A_s commute on each V_{λ_i}, therefore commute. □

Proof of the proposition. Consider a Jordan decomposition $A = A'_s + A'_n$, and let $A = A_s + A_n$ be the classical Jordan decomposition. Take the difference, we get

$$A_s - A'_s = A_n - A'_n$$

But A'_s commutes with A'_n and itself, hence with A. Hence by taking $B = A'_s$ in lemma (b), we conclude that A'_s commutes with A_s and A_n. Therefore $A'_n = A - A'_s$ also commutes with A_s and A_n. So in (2) we we have difference of commutative operators on both sides. Hence LHS is diagonalizable and RHS is nilpotent (by the binomial formula). But equality of a diagonalizable operator to a nilpotent one is possible only if both are 0.

Question. Is it true in general that Jordan decomposition is unique?

Exercise 6.1. Show that any nonabelian 3-dimensional nilpotent Lie algebra is isomorphic to the Heisenberg algebra H_3.

Proof. If $\mathfrak{g}$ is nonabelian and 3-dimensional, then $Z(\mathfrak{g})$ must have dimension less than 3. By a previous exercise (3.2), $\dim Z(\mathfrak{g}) \neq \dim \mathfrak{g} - 1$, so this dimension cannot be 2. A proposition from lecture 4 states that if $\mathfrak{g}$ is nonzero and nilpotent, $Z(\mathfrak{g})$ is nonzero. Hence $Z(\mathfrak{g})$ is 1-dimensional.

Now by exercise 3.3, the n-dimensional Lie algebras for which $Z(\mathfrak{g})$ has dimension two less than $\mathfrak{g}$ are $Ab_{n-2} \oplus \mathfrak{g}_2$ and $Ab_{n-3} \oplus H_3$, where $\mathfrak{g}_2$ is the 2-dimensional Lie algebra $\mathbb{F}x + \mathbb{F}y$ defined by $[x, y] = y$.

Since $\mathfrak{g}$ is nilpotent, it cannot be $Ab_1 \oplus \mathfrak{g}_2$, because $\mathfrak{g}_2$ is not nilpotent. Then the only remaining possibility is $\mathfrak{g} = Ab_0 \oplus H_3 = H_3$. □

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and π its representation on a finite-dimensional vector space V, over an algebraically closed field $\mathbb{F}$ of characteristic 0. We have the following generalized eigenspace decompositions for a fixed $a \in \mathfrak{g}$.

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda^a \qquad V_\lambda^a = \left\{ v \in V \mid (\pi(a) - \lambda I)^N v = 0 \text{ for some } N \in \mathbb{N} \right\}$$

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{F}} \mathfrak{g}_\alpha^a \qquad \mathfrak{g}_\alpha^a = \left\{ g \in \mathfrak{g} \mid (\mathbf{ad}\ a - \alpha I)^N g = 0 \text{ for some } N \in \mathbb{N} \right\}$$

We'll prove the following.

Theorem 6.4. $\pi\left(\mathfrak{g}_\alpha^a\right) V_\lambda^a \subseteq V_{\lambda+\alpha}^a$

First, we need a lemma on associative algebras.

Lemma 6.5. *Suppose U is a unital associative algebra over $\mathbb{F}$, and let $a, b \in U$ and $\lambda, \alpha \in \mathbb{F}$. Then*

$$(a-\alpha-\lambda)^N b = \sum_{j=0}^{N} \binom{N}{j} \left((\mathbf{ad}\ a - \alpha I)^j\, b\right)(a-\lambda)^{N-j}.$$

Proof. Write $\mathbf{ad}\ a = L_a - R_a$, where $L_a(x) = ax$ and $R_a(x) = xa$. Then

$$\begin{aligned} L_{a-\alpha-\lambda} &= L_a - \alpha I - \lambda I \\ &= \mathbf{ad}\ a + R_a - \alpha I - \lambda I \\ L_{a-\alpha-\lambda} &= (\mathbf{ad}\ a - \alpha) + R_{a-\lambda} \end{aligned} \tag{1}$$

For any given $a, b \in U$, the operators L_a and R_b commute by associativity of U. Since $\mathbf{ad}\ a$ is just the difference $L_a - R_a$, it commutes with both L_a and R_a. Then since $\alpha I, \lambda I \in \mathbb{F}I \subset Z(U)$, the terms $(\mathbf{ad}\ a - \alpha)$ and $R_{a-\lambda}$ on the right side of (1) commute. Given this, the claimed equality follows from raising both sides of (1) to the Nth power and applying the Binomial Theorem. □

Proof of Theorem 6.4. Applying the lemma to $\pi(\mathfrak{g})$, we have the following for all $g \in \mathfrak{g}$, and thus for all $g \in \mathfrak{g}_\alpha^a$. (Recall that $a \in \mathfrak{g}$ is fixed.)

$$(\pi(a)-\alpha-\lambda)^N \pi(g) = \sum_{j=0}^{N} \binom{N}{j} (\mathbf{ad}\ \pi(a)-\alpha)^j\, \pi(g)\, (\pi(a)-\lambda)^{N-j}$$

Apply both sides of this to $v \in V_\lambda^a$ with $N > \dim V_\lambda^a + \dim \mathfrak{g}_\alpha^a$. By this choice of N, either $j > \dim \mathfrak{g}_\alpha^a$ or $N - j > \dim V_\lambda^a$. If $j > \dim g_\alpha^a$, then $(\mathbf{ad}\ \pi(a)-\alpha)^j\, \pi(g) = 0$ since $g \in g_\lambda^a$. Otherwise, $N - j > \dim V_\lambda^a$, so $(\pi(a)-\lambda)^{N-j}\, v = 0$ since $v \in V_\lambda^a$.

This makes every term in the sum on the right zero, so $(\pi(a)-\alpha-\lambda)^N \pi(g)v = 0$. Then $\pi(g)v$ is a generalized eigenvector of $\pi(a)$ with eigenvalue $\alpha - \lambda$, so $\pi(g)v \in V_{\lambda+\alpha}^a$. Since this holds for all $g \in \mathfrak{g}_\alpha^a$ and $v \in V_\lambda^a$, the claimed inclusion holds. □

By analogy to the definition of a generalized eigenspace, we can define generalized weight spaces of a Lie algebra $\mathfrak{g}$.

Definition 6.3. Let $\mathfrak{g}$ be a Lie algebra with a representation π on a vector space on V, and let $\lambda \in \mathfrak{g}^*$ be a linear functional on $\mathfrak{g}$. The generalized weight space of $\mathfrak{g}$ in V attached to λ is

$$V_\lambda^{\mathfrak{g}} = \left\{ v \in V \mid (\pi(g)-\lambda(g)I)^N\, v = 0 \text{ for some } N \text{ depending on } g,\ \text{for all } g \in \mathfrak{g} \right\}.$$

Under the right conditions, a nilpotent subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ permits decomposing V as a direct sum of the generalized weight spaces of $\mathfrak{h}$, each of which is a subrepresentation of $\pi_{\mathfrak{h}}$. The following theorem makes this precise.

Theorem 6.6. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and π its representation on a finite-dimensional vector space V, over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $\mathfrak{h}$ be a nilpotent subalgebra of $\mathfrak{g}$. Then the following equalities hold.*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{\mathfrak{h}} \tag{2}$$

$$\pi\left(\mathfrak{g}_\alpha^{\mathfrak{h}}\right) V_\lambda^{\mathfrak{h}} \subseteq V_{\lambda+\alpha}^{\mathfrak{h}} \tag{3}$$

Remark. In the case of the adjoint representation, we may express these as follows.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha^{\mathfrak{h}} \tag{4}$$

$$\left[\mathfrak{g}_\alpha^{\mathfrak{h}}, \mathfrak{g}_\beta^{\mathfrak{h}}\right] \subseteq \mathfrak{g}_{\alpha+\beta}^{\mathfrak{h}} \tag{5}$$

Proof of Theorem 6.6.

Case 1. For each $a \in \mathfrak{h}$, $\pi(a)$ has only one eigenvalue.

In this case, V is a generalized eigenspace $V_{\lambda(a)}^a$ of every $a \in \mathfrak{h}$, so we just need to check the linearity of λ.

Since $\mathfrak{h}$ is nilpotent, it is solvable. Since we assumed $\mathbb{F}$ to be algebraically closed and with characteristic 0, we can then apply Lie's theorem, which guarantees the existence of a weight λ' with some nonzero weight space $V_{\lambda'}^{\mathfrak{h}}$. Then $\lambda'(a)$ must be the eigenvalue of $\pi(a)$ with which $\pi(a)$ acts on $V_{\lambda'}^{\mathfrak{h}}$, so $\lambda' = \lambda$. Therefore λ is linear, so V is the generalized weight space $V_\lambda^{\mathfrak{h}}$.

Case 2. For some $a_0 \in \mathfrak{h}$, $\pi(a_0)$ has at least two distinct eigenvalues.

Since $\mathfrak{h}$ is nilpotent, **ad** a is a nilpotent operator on $\mathfrak{h}$ for all $a \in \mathfrak{h}$. Thus $\mathfrak{h} \subset \mathfrak{g}_0^a$. Then by Theorem 6.4, $\pi(\mathfrak{h})V_\lambda^a \subseteq V_\lambda^a$ for any $a \in \mathfrak{h}$.

Since $\mathbb{F}$ is algebraically closed, V can be written as a direct sum of the generalized eigenspaces of a_0. Since each $V_\lambda^{a_0}$ is invariant under the action of $\mathfrak{h}$, each $V_\lambda^{a_0}$ is also a representation of $\mathfrak{h}$. Since $\dim V_\lambda^{a_0} < \dim V$, we may apply induction on $\dim V$. This establishes the equality (2).

To finish, we'll prove the inclusion (3). Suppose $\alpha, \lambda \in \mathfrak{h}^*$, and suppose $g \in \mathfrak{g}_\alpha^{\mathfrak{h}}$. Then $g \in \mathfrak{g}_{\alpha(a)}^a$ for all $a \in \mathfrak{h}$. By Theorem 6.4, $\pi(g)V_{\lambda(a)}^a \subset V_{\lambda(a)+\alpha(a)}^a$ for all $a \in \mathfrak{h}$. Then

$$v \in \bigcap_{a \in \mathfrak{h}} V_{\lambda(a)}^a \implies \pi(g)v \in \bigcap_{a \in \mathfrak{h}} V_{\lambda(a)+\alpha(a)}^a.$$

Since $\bigcap_{a \in \mathfrak{h}} V_{\lambda(a)}^a = V_\lambda^{\mathfrak{h}}$ by the definition of a generalized weight space, this establishes (3). □

Exercise 6.2. Suppose $\mathbb{F}$ has characteristic 2, and $V = \mathbb{F}[x]/(x^2)$ is a representation of H_3 where $p \mapsto \frac{\partial}{\partial x}$, $q \mapsto x$, and $c \mapsto I$. Then $V = V_\lambda$, but λ is not a linear function on H_3. Compute λ.

Proof. Suppose p acts as $\frac{\partial}{\partial x}$, q acts as multiplication by x, and c acts as the identity on $\mathbb{F}[x]/(x^2)$. Then:

$$\begin{aligned} p(a+bx) &= b + 0x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ q(a+bx) &= 0 + ax = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ c(a+bx) &= a + bx = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

Then making a basis on $\mathbb{F}[x]/(x^2)$ using 1 and x, we can write the matrix representing some $rp + sq + tc \in H_3$ as the following matrix.

$$\begin{bmatrix} t & r \\ s & t \end{bmatrix}$$

Then finding λ is a matter of solving its characteristic polynomial.

$$\begin{aligned} 0 &= \det \begin{bmatrix} t-\lambda & r \\ s & t-\lambda \end{bmatrix} \\ &= (t-\lambda)^2 - rs \\ \pm\sqrt{rs} &= t - \lambda \\ \lambda &= t \pm \sqrt{rs} \end{aligned}$$

In a field of characteristic 2, we can drop the $\pm$ sign. By passing to the algebraic closure if necessary, we can assume the square root of rs always exists. Thus:

$$\lambda(rp + sq + tc) = t + \sqrt{rs}$$

(To verify that λ is not linear, observe that by this formula, $\lambda(p) = \lambda(q) = 0$, but $\lambda(p+q) = 1$.) $\square$

Exercise 6.3. By the example of the adjoint representation of a nonabelian solvable Lie algebra, show that the generalized weight space decomposition fails if the Lie algebra is solvable but not nilpotent.

Proof. Consider the Lie algebra $\mathfrak{g}_2 = \mathbb{F}x + \mathbb{F}y$, with the bracket operation defined by $[x, y] = y$.

It's apparent by induction that $\mathfrak{g}_2^k = [\mathfrak{g}_2, \mathfrak{g}_2^{k-1}] = \mathbb{F}y$ (for $k \geq 2$), so $\mathfrak{g}_2$ is not nilpotent. However, then $\mathfrak{g}_2^{(1)} = \mathbb{F}y$, which is 1-dimensional, so $\mathfrak{g}_2^{(2)} = 0$, and thus $\mathfrak{g}_2$ is solvable.

Taking x and y as the basis elements of $\mathfrak{g}_2$, the adjoint representation takes x and y to the following matrices.

$$x \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad y \mapsto \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

So we find their eigenvalues by solving their characteristic polynomials.

$$\begin{aligned} \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \text{ or } 1 \end{aligned} \qquad\qquad \begin{aligned} \lambda^2 &= 0 \\ \lambda &= 0 \end{aligned}$$

The corresponding generalized eigenvectors can be found by lucky guessing. Specifically, **ad** x has x with eigenvalue 0 and y with eigenvalue 1, while **ad** y has all of $\mathfrak{g}_2$ with eigenvalue 0.

So we can get weight spaces $V_0 = \text{span}\{x\}$ and $V_{x^*} = \text{span}\{y\}$, corresponding to the zero linear functional and the linear functional defined by $x \mapsto 1$. The vector space decomposes into the direct sum of these weight spaces, but the *representation* does not! Specifically, V_0 is not closed under the action of y. □

Exercise 6.4. Take $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{h} = \{\text{diagonal matrices}\}$. Find the generalized weight space decomposition in both the tautological and the adjoint representations, and check the inclusions (3) and (5) in Theorem 6.6.

Proof. Suppose $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{h} \subset \mathfrak{g}$ consists of diagonal matrices. Then the generalized eigenvectors of $h \in \mathfrak{h}$ are actual eigenvectors, so every standard basis element of $\mathbb{F}^n$ is an eigenvector. Also, given any linear combination $ae_i + be_j$ of more than one basis element, there is some diagonal matrix that takes e_i to e_i and e_j to zero, so these linear combinations are not generalized eigenvectors of everything in $\mathfrak{h}$. Thus the only candidates for generalized weight spaces are the n axes, each of which is the span of a single standard basis element of $\mathbb{F}^n$.

For the axis V_i spanned by e_i, the linear functional on $\mathfrak{h}$ that takes h to the component $h_{i,i}$ is a weight making V_i a weight space. Thus in the tautological representation, $\mathbb{F}^n$ decomposes as a direct sum of n copies of $\mathbb{F}$.

To do the same with the adjoint representation, suppose the diagonal entries of $h \in \mathfrak{h}$ are h_i. Then for $a \in \mathfrak{g}$, we have:

$$\begin{aligned}((\mathbf{ad}\ h)a)_{i,j} &= (ha - ah)_{i,j} \\ &= h_i a_{i,j} - a_{i,j} h_j \\ &= (h_i - h_j) a_{i,j}\end{aligned}$$

This shows that **ad** h is diagonalizable, which again implies that its generalized eigenvectors are actual eigenvectors, and so its generalized weight spaces are actually weight spaces.

Possible pairs of eigenvalues are

1. $h_i - h_j$ vs. $h_i - h_k$,
2. $h_i - h_j$ vs. $h_k - h_\ell$,
3. $h_i - h_j$ vs. $h_j - h_i$, and
4. $h_i - h_i$ vs. $h_j - h_j$.

By appropriate choice of h, we can always make distinct eigenvalues in (1) and (2), so the basis elements $e_{i,j}$ of $\mathfrak{g}$ satisfying $i < j$ lie in distinct eigenspaces for some h, and thus they lie in distinct candidate weight spaces. This weight space can be achieved with the linear functional $\lambda_{i,j}$ taking h to $h_i - h_j$.

Since for the theorem we assume the characteristic of $\mathbb{F}$ is not 2, the eigenvalues in (3) will be distinct, so we'll also have $\lambda_{i,j}$ with $i > j$. Finally, both eigenvalues in (4) are always zero, so the zero linear functional has $\mathfrak{h}$ as its weight space.

Combining all of this, the generalized weight space decomposition of $\mathfrak{g}$ in the adjoint representation is $\mathfrak{h}$ plus some 1-dimensional weight spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \operatorname{span}\{e_{i,j}\}$$

To check the assertion of the theorem from class, first we verify for the tautological representation that:

$$\pi\left(\mathfrak{g}_\alpha^\mathfrak{h}\right) V_\lambda^\mathfrak{h} \subseteq V_{\lambda+\alpha}^\mathfrak{h}$$

Each $V_\lambda^\mathfrak{h}$ is the span of some basis element e_i, with λ corresponding to the map $h \mapsto h_i$, so we really only need to check that, for some appropriate j:

$$\pi\left(\mathfrak{g}_\alpha^\mathfrak{h}\right) e_i \propto e_j$$

In the case $\alpha = 0$ we should get $j = i$, and we do; the space $\mathfrak{g}_0^\mathfrak{h}$ consists of all diagonal matrices, so they act on e_i by scaling.

In the case $\alpha h = h_k - h_\ell$, we then have $\mathfrak{g}_\alpha^\mathfrak{h} = e_{k,\ell}$, and $\alpha + \lambda = h_k - h_\ell + h_i$. We should expect zero if $i \neq \ell$, since we only have nonzero weight spaces for λ of the form $h \mapsto h_{something}$; and indeed this is the case, since if $i \neq \ell$ then $e_{k,\ell} e_i = 0$. Furthermore, if $i = \ell$ then we should get the span of e_k, which is the weight space corresponding to $h \mapsto h_k$. We verify this by observing that $e_{k,i} e_i = e_k$. So in fact we have equality:

$$\pi\left(\mathfrak{g}_\alpha^\mathfrak{h}\right) V_\lambda^\mathfrak{h} = V_{\lambda+\alpha}^\mathfrak{h}$$

The second assertion in (b) of the theorem is essentially the same statement for the adjoint representation. First, if $\alpha = \beta = 0$, then both $\mathfrak{g}_\alpha^\mathfrak{h}$ and $\mathfrak{g}_\beta^\mathfrak{h}$ are equal to $\mathfrak{h}$. Since $\mathfrak{h}$ consists of diagonal matrices, it's commutative, so the bracket is zero and thus is contained in any weight space we like.

If $\alpha = 0$ and $\beta = \{h \mapsto h_i - h_j\}$, then we end up with $[\mathfrak{h}, \operatorname{span}\{e_{i,j}\}]$. Observe that:

$$\begin{aligned} [h, e_{i,j}] &= h e_{i,j} - e_{i,j} h \\ &= (h_i - h_j) e_{i,j} \in \operatorname{span}\{e_{i,j}\} \end{aligned}$$

Finally, if α maps h to $h_i - h_j$ and β maps h to $h_k - h_\ell$, we have essentially $[e_{i,j}, e_{k,\ell}]$. We need either $j = k$ or $i = \ell$ for $\alpha + \beta$ to be a weight, and we indeed see that if neither holds, then $e_{i,j} e_{k,\ell} = 0$. Otherwise, by relabeling α and β, we can assume without loss of generality that $j = k$. This gives us $e_{i,\ell}$ if $i \neq \ell$ and 0 if $i = \ell$, so either way it's in the weight space of $h \mapsto h_i - h_\ell$. □