**18.745 Introduction to Lie Algebras** September 23, 2010

Lecture 5 — Lie's Theorem

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# 1 Lie's Theorem

## 1.1 Weight spaces

**Notation 1.1.** Let $V$ be a vector space over a field $\mathbb{F}$. We will denote by $V^*$ the dual vector space.

**Definition 1.1.** Let $\mathfrak{h}$ be a Lie algebra, $\pi : \mathfrak{h} \to \mathfrak{gl}_V$ a representation of $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. We define the weight space of $\mathfrak{h}$ attached to $\lambda$ as

$$V_\lambda^{\mathfrak{h}} = \{v \in V \mid \pi(h)v = \lambda(h)v, \ \forall h \in \mathfrak{h}\}.$$

If $V_\lambda^{\mathfrak{h}} \neq 0$, we will say that $\lambda$ is a weight for $\pi$.

## 1.2 Lie's Lemma

**Lemma 1.1** (Lie's Lemma)**.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an ideal both over $\mathbb{F}$ algebrically closed and of characteristic 0. Let $\pi$ be a representation of $\mathfrak{g}$ in a finite dimensional $\mathbb{F}-$vector space $V$. Then each weight space $V_\lambda^{\mathfrak{h}}$ for the restricted representation $\pi_{|\mathfrak{h}}$ is invariant under $\mathfrak{g}$.*

*Proof.* We wish to show that if $v \in V_\lambda^{\mathfrak{h}}$, then $\pi(a)v \in V_\lambda^{\mathfrak{h}}$, $\forall a \in \mathfrak{g}$. But this is verified if and only if

$$\pi(h)\pi(a)v = \lambda(h)\pi(a)v, \ \forall h \in \mathfrak{h}, \ \forall a \in \mathfrak{g}. \tag{1}$$

Now,

$$\pi(h)\pi(a)v = [\pi(h), \pi(a)]v + \pi(a)\pi(h)v = \pi([h, a])v + \pi(a)\lambda(h)v. \tag{2}$$

Since $\mathfrak{h}$ is an ideal, $[h, a] \in \mathfrak{h}$, then (2) becomes

$$\pi(h)\pi(a)v = \lambda([h, a])v + \pi(a)\lambda(h)v. \tag{3}$$

Thus, it is sufficient to show that

$$\lambda([h, a]) = 0, \forall h \in \mathfrak{h},$$

whenever $V_\lambda^{\mathfrak{h}} \neq 0$.
Let us fix $a \in \mathfrak{g}$ and let $0 \neq v$ be an element in $V_\lambda^{\mathfrak{h}} \neq \{0\}$. We define

$$W_m = span \ < v, \pi(a)v, \pi^2(a)v, \ldots, \pi^m(a)v >, \ \forall m \geq 0, \ W_{-1} = \{0\}.$$

Since $V$ is finite dimensional, then there exists $N \in \mathbb{N}$ s.t. $N$ is the maximal integer for which all the generators of $W_N$ are linearly indipendent. Then we have $W_N = W_{N+1} = \ldots$, hence

$\pi(a)W_N \subset W_N$.
We will consider the increasing sequence of subspaces

$$W_{-1} = \{0\} \subset W_0 = \{\mathbb{F}v\} \subset \cdots \subset W_N.$$

Now we claim that $\forall m \geq 0,\ W_m$ is invariant under $\pi(\mathfrak{h})$ and furthermore

$$\forall h \in \mathfrak{h},\ \pi(h)\pi(a)^m v - \lambda(h)\pi(a)^m v \in W_{m-1}. \tag{4}$$

We prove (4) by induction on $m$. The case $m = 0$ is true since $v \in V_\lambda^{\mathfrak{h}}$.
Suppose that we have proved the assumption for $m-1$: we want to prove it for $m$.

$$\begin{aligned} & \pi(h)\pi(a)^m v - \lambda(h)\pi(a)^m v = [\pi(h), \pi(a)]\pi(a)^{m-1} v + \pi(a)\pi(h)\pi(a)^{m-1} v - \lambda(h)\pi(a)^m v = \\ = \ & [\pi(h), \pi(a)]\pi(a)^{m-1} v + \pi(a)\pi(h)\pi(a)^{m-1} v - \pi(a)\lambda(h)\pi(a)^{m-1} v. \end{aligned}$$

By induction hypothesis, we have that

$$w = \pi(h)\pi(a)^{m-1} v - \lambda(h)\pi(a)^{m-1} v \in W_{m-2}$$

and $\pi(a)w \in W_{m-1}$, by construction of the $W_i$'s.
Moreover, $\mathfrak{h}$ is an ideal so that $[\pi(h), \pi(a)] \in \pi(\mathfrak{h})$ and by inductive hypothesis

$$[\pi(h), \pi(a)]\pi(a)^{m-1} v \in W_{m-1},$$

thus,

$$\pi(h)\pi(a)^m v - \lambda(h)\pi(a)^m v \in W_{m-1}$$

because it is a sum of elements in $W_{m-1}$. This concludes the proof of the inductive step.
We know that $W_N$ is invariant both for $\pi(a)$ and for $\pi(h)$, $\forall h \in \mathfrak{h}$. In particular,(4) shows that $\forall h \in \mathfrak{h},\ \pi(h)$ acts on $W_N$ as an upper triangular matrix, with in the basis $\{v, \pi(a)v, \ldots, \pi(a^N)v\}$,

$$\begin{pmatrix} \lambda(h) & * & \ldots & * & * \\ 0 & \ddots & * & \ldots & \vdots \\ 0 & 0 & \lambda(h) & * & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda(h) \end{pmatrix}$$

As a consequence of this, we have that

$$tr_{W_N}([\pi(h), \pi(a)]) = 0 = tr_{W_N}(\pi[h, a]) = N\lambda([h, a])$$

which implies that $\lambda([h, a]) = 0$, since $char\ \mathbb{F} = 0$. This concludes the proof. $\square$

**Lie's Theorem.** *Let $\mathfrak{g}$ be a solvable Lie algebra and $\pi$ a representation of $\mathfrak{g}$ on a finite dimensional vector space $V \neq 0$, over an algebraically closed field $\mathbb{F}$ of characteristic $0$. Then there exists a weight $\lambda \in \mathfrak{g}^*$ for $\pi$, that is $V_\lambda^{\mathfrak{g}} \neq \{0\}$.*

*Proof.* We can suppose that $\mathfrak{g}$ is finite dimensional, since $V$ is finite dimensional and the representation factors in the following way:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\pi"] \arrow[dr, "\phi"'] & & End(V) \\ & \pi(\mathfrak{g}) \arrow[ur, "i"'] & \end{tikzcd}$$

As $\mathfrak{g}$ is solvable, also $\pi(\mathfrak{g})$ will be solvable. This follows from the fact that $\pi(\mathfrak{g})^{(n)} = \pi(\mathfrak{g}^{(n)})$ - by induction on $n$.

We will prove Lie's theorem by induction on the dimension of $\mathfrak{g}$, $\dim \mathfrak{g} = m$.

The case $\dim \mathfrak{g} = 0$ is trivial.

Suppose now that we have proved Lie's theorem $\dim \mathfrak{g} = m - 1$, we want to show that the theorem holds also for $\mathfrak{g}$, $\dim \mathfrak{g} = m \geq 1$.

Since $\mathfrak{g}$ is solvable, of positive dimension, $\mathfrak{g}$ properly includes $[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, any subspace is automatically an ideal.

Take a subspace of codimension one in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, then its inverse image $\mathfrak{h}$ is an ideal of codimension one in $\mathfrak{g}$ (including $[\mathfrak{g}, \mathfrak{g}]$). Thus, we have the following decomposition of $\mathfrak{g}$ as a vector space

$$\mathfrak{g} = \mathfrak{h} + \mathbb{F}a.$$

Now, $\dim \mathfrak{h} = m - 1$ and it is solvable (since an ideal of a solvable Lie algebra is solvable), hence by inductive hypothesis we can find a non-zero weight space $V_\lambda^{\mathfrak{h}} \neq \{0\}$, $\lambda \in \mathfrak{h}^*$. By Lie's lemma, $V_\lambda^{\mathfrak{h}}$ is invariant under the action of $\pi(\mathfrak{g})$. In particular, $\pi(a)V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}$, hence (since $F$ is algebraically closed) there exists $0 \neq v \in V_\lambda^{\mathfrak{h}}$ such that $\pi(a)v = la$, for some $l \in \mathbb{F}$. We define a linear functional $\lambda' \in \mathfrak{g}^*$ on $\mathfrak{g}$ by

$$\lambda'(h + \mu a) = \lambda(h) + \mu l, \ \forall h \in \mathfrak{h}, \ l \in \mathbb{F}.$$

Thus, by construction, we see that $v$ belongs to $V_{\lambda'}^{\mathfrak{g}}$. In particular $V_{\lambda'}^{\mathfrak{g}} \neq \{0\}$. □

**Exercise. 5.1** Show that we may relax the assumption on $\mathbb{F}$ in Lie's Lemma. Show Lie's Lemma under the assumptions that $\mathbb{F}$ is algebraically closed and that $\dim V < char\, \mathbb{F}$.

In the proof, we calculated the trace of a certain endomorphism of $W \subseteq V$ to be zero, and also to be $Nq$, where $N = \dim W$ and $q$ was a quantity that we needed to show was zero. Of course then if the characteristic of $\mathbb{F}$ exceeds the dimension of $V$, $\dim W$ is non-zero in $\mathbb{F}$ and we may conclude $q = 0$. Given this weaker assumption, the rest goes through unaltered.

**Exercise. 5.2** Consider the Heisenberg algebra $H_3$ and its representation on $\mathbb{F}[x]$ given by

$$\begin{aligned} c &\mapsto Id, \\ p &\mapsto (f(x) \mapsto xf(x)), \ \forall f(x) \in \mathbb{F}[x], \\ q &\mapsto (f(x) \mapsto \frac{d}{dx} f(x)), \ \forall f(x) \in \mathbb{F}[x]. \end{aligned}$$

Show that the ideal generated by $x^N$, $0 < N = char\, \mathbb{F}$, in $\mathbb{F}[x]$ is invariant for the representation of $H_3$ and that the induced representation of $H_3$ on $\mathbb{F}[x]/(x^N)$ has no weight.

$(x^N)$ is a subrepresentation, as it is preserved by the action of $p$ and $c$, and the action of $q$ annihilates the $x^N$ term when differentiating. Now consider $v = a_0x_0 + a_1x_1 + \ldots + a_{N-1}x^{N-1}$, a representative of an element of the quotient. All members of the quotient will be represented this way. Now if $v$ is a weight vector, its derivative must be $kv$ for some $k$, so that $ia_i = ka_{i-1}$ for each $i < N$. On the other hand, in order that it's an eigenvector for $p$, it must have zero constant term, or have $pv = 0$. These two statements show easily that $v = 0$, (as $i \neq 0$ when $i = 1, \ldots, N-1$). Thus there is no weight vector.

**Exercise. 5.3** Show the following two corollaries to Lie's Theorem:

- for all representations $\pi$ of a solvable Lie algebra $\mathfrak{g}$ on a finite dimensional vector space $V$ over an algebraically closed field $\mathbb{F}$, *char* $\mathbb{F} = 0$, there exists a basis for $V$ for which the matrices of $\pi(\mathfrak{g})$ are upper triangular;
- a solvable subalgebra $\mathfrak{g} \subset \mathfrak{gl}_V$ ($V$ is finite dimensional over an algebraically closed field $\mathbb{F}$, *char* $\mathbb{F} = 0$) is contained in the subalgebra of upper triangular matrices over $\mathbb{F}$ for some basis of $V$.

The second statement is simply an application of the first. We prove the first by induction on the module's dimension. It is trivial in dimension 1. Suppose $V$ is a module. Use Lie's theorem to find a weight $v$ of $V$. The quotient module $V/\mathbb{F}v$, by induction, can given a basis such that $\mathfrak{g}$ acts by upper triangular matrices. Taking any preimages of that basis in $V$, and extending it to a basis of $V$ by including $v$, we obtain a basis of $V$ on which $\mathfrak{g}$ acts by upper triangular matrices.

**Exercise. 5.4** Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over the algebrically closed field $\mathbb{F}$, *char* $\mathbb{F} = 0$. Show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

If $[\mathfrak{g}/Z(\mathfrak{g}), \mathfrak{g}/Z(\mathfrak{g})]$ is nilpotent, so is $[\mathfrak{g}, \mathfrak{g}]$, so using the adjoint representation, we may assume $\mathfrak{g}$ is a solvable subalgebra of $\mathfrak{gl}_{\mathfrak{g}}$. Then by the previous, there is a basis in which $\mathfrak{g}$ consists of upper triangular matrices. Then $[\mathfrak{g}, \mathfrak{g}]$ consists of strictly upper triangular matrices, and is a subalgebra of the nilpotent Lie algebras $\mathfrak{u}_n$.