18.745 Introduction to Lie Algebras	September 21, 2010

Lecture 4 — Nilpotent and Solvable Lie Algebras

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4.1 Preliminary Definitions and Examples

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The *lower central series* of $\mathfrak{g}$ is the descending chain of subspaces

$$\mathfrak{g}^1 = \mathfrak{g} \supseteq \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] \supseteq \mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2] \supseteq \ldots \supseteq \mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}] \supseteq \ldots$$

while the *derived series* is

$$\mathfrak{g}^{(0)} = \mathfrak{g} \supseteq \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \supseteq \ldots \supseteq \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \supseteq \ldots$$

We note that

(1) $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$ for $n \geq 1$ by induction

(2) All $\mathfrak{g}^n$ and $\mathfrak{g}^{(n)}$ are ideals in $\mathfrak{g}$

Definition 4.2. A lie algebra $\mathfrak{g}$ is called *nilpotent* (resp. *solvable*) if $\mathfrak{g}^n = 0$ for some $n > 0$ (resp. $\mathfrak{g}^{(n)} = 0$ for some $n > 0$).

If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}$ is solvable. In fact

$$\{\text{abelian}\} \subsetneq \{\text{nilpotent}\} \subsetneq \{\text{solvable}\}$$

Example 4.1. Let $\mathfrak{g} = \mathbb{F}a + \mathbb{F}b$ with $[a, b] = b$, $\mathfrak{g}^{(1)} = \mathfrak{g}^2 = \mathbb{F}b$, $\mathfrak{g}^3 = \mathfrak{g}^4 = \ldots = \mathbb{F}b$ but $\mathfrak{g}^{(2)} = 0$ so $\mathfrak{g}$ is solvable but not nilpotent.

Example 4.2. Let $H_3 = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c$ with $[c, \mathfrak{g}] = 0$ and $[p, q] = c$. Then $H_3^2 = \mathbb{F}c, H_3^3 = 0$.

Example 4.3.

$$\begin{aligned} gl_n(\mathbb{F}) &\supseteq b_n = \{\text{upper triangular matrices}\} \\ &\supseteq \eta_n = \{\text{strictly upper triangular matrices}\} \end{aligned}$$

Exercise 4.1. Show b_n is a solvable (but not nilpotent) Lie algebra and that $[b_n, b_n] = \eta_n (n \geq 2)$. Also show that η_n is a nilpotent Lie algebra.

Proof. Consider $C = AB - BA$ for $A, B \in b_n$. Say $A = (a_{ij}), B = (b_{ij})$, and $C = (c_{ij})$. Then

$$c_{ij} = \sum_{k=1}^{n} (a_{ik}b_{kj} - b_{ik}a_{kj})$$

We notice $a_{ik} = b_{ik} = 0$ if $k < i$ and $b_{kj} = a_{kj} = 0$ if $k > j$. Thus

$$c_{ij} = \sum_{k=i}^{j} (a_{ik} b_{kj} - b_{ik} b_{kj})$$

Then if $i = j$, $c_{ii} = a_{ii} b_{ii} - b_{ii} a_{ii} = 0$, implying $C \in \eta_n$, i.e., $[b_n, b_n] \in \eta_n$. Define the kth diagonal to be the set of c_{ij} where $j - i = k$. We show that the kth diagonal contains no nonzero entries in b_n^{k+1}. We have shown this to be true for $k = 0$. Now assume it is true up to $k = m$ for some $m \geq 0$. Then any $C = [A, B] \in b_n^{m+1}$ is such that $A, B \in b_n^m$ and thus

$$c_{i,i+m} = \sum_{k=1}^{i+m} (a_{ik} b_{k,i+m} - b_{ik} a_{k,i+m})$$

Now $a_{jk} \neq 0$, $b_{jk} \neq 0$ implies $k = i + m$ while $a_{k,i+m} \neq 0$, $b_{k,i+m} \neq 0$ implies $k = i$ but $m \neq 0$ implies $c_{i,i+m} = 0$ which is equivalent to saying all diagonals are zero so b_n is solvable. Next we show $[b_n, b_n] \supseteq \eta_n$ so that we finally know $[b_n, b_n] = \eta_n$. Consider the basis element $e_{ij} (j > i)$ which is defined to have entry (i, j) equal to 1, and all other entires 0. Then $[e_{ij}, e_{jj}] = e_{ij} e_{jj} - e_{jj} e_{ij} = e_{ij} e_{jj} = e_{ij} \Rightarrow e_{ij} \in [b_n, b_n] \; \forall e_{ij}$ such that $j > i \Rightarrow [b_n, b_n] \supseteq \eta_n$ This shows that b_n is not nilpotent. □

4.2 Simple Facts about Nilpotent and Solvable Lie Algebras

First we note

1. Any subalgebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable).
2. Any factor algebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable)

Exercise 4.2. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be an ideal. Show that if $\mathfrak{h}$ is solvable and $\mathfrak{g}/\mathfrak{h}$ is solvable, then $\mathfrak{g}$ is solvable too.

Proof. First we prove that all the homomorphic images of a solvable algebra are solvable. Let $\mathfrak{g}_1$ be solvable and $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ a surjective homomorphism. We show

$$\phi(g_1^{(i)}) = \mathfrak{g}_2^{(i)}$$

The case $i = 0$ is trivial. Suppose it holds for some $i \geq 0$. Then

$$\begin{aligned} \phi(\mathfrak{g}_1^{(i+1)}) &\supseteq \phi([\mathfrak{g}_1^{(i)}, \mathfrak{g}_1^{(i)}]) \\ &= [\phi(\mathfrak{g}_1^{(i)}), \phi(\mathfrak{g}_1^{(i)})] \\ &= [\mathfrak{g}_2^{(i)}, \mathfrak{g}_2^{(i)}] \\ &= \mathfrak{g}_2^{(i+1)} \end{aligned}$$

Thus if $\mathfrak{g}_1$ is solvable, so is $\mathfrak{g}_2$. Now suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a solvable ideal, say $\mathfrak{h}^{(n)} = 0$, and $(\mathfrak{g}/\mathfrak{h})^{(m)} = 0$. Consider the canonical homomorphism

$\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ From the previous result,

$$\pi(\mathfrak{g}^{(m)}) = (\mathfrak{g}/\mathfrak{h})^{(m)} = 0 \Rightarrow \mathfrak{g}^{(m)} \subseteq I$$

Then $(\mathfrak{g}^{(m)})^{(n)} = \mathfrak{g}^{(m+n)} \subseteq I^{(n)} = 0$ which means that $\mathfrak{g}$ is solvable. □

The last exercise does not hold if we everywhere put "nilpotent" in place of "solvable," as the following example shows.

Example 4.4. Suppose g= Fa+Fb, $[a, b] = b$. $\mathbb{F}b \subset \mathfrak{g}$ is an ideal, $\mathbb{F}b$ and $\mathfrak{g}/\mathbb{F}b$ are 1-dimensional and hence abelian and nilpotent. But $\mathfrak{g}$ is not nilpotent.

Theorem 4.1. *(a) If $\mathfrak{g}$ is a nonzero nilpotent Lie algebra then $Z(\mathfrak{g})$ is nonzero*

(b) If $\mathfrak{g}$ is a finite-dimensional Lie algebra such that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent.

Proof. (a) Take $N > 0$ minimal such that $\mathfrak{g}^N = 0$. Since $\mathfrak{g} \neq 0$, $N \geq 2$, but then $\mathfrak{g}^{N-1} \neq 0$ and $[\mathfrak{g}, \mathfrak{g}^{N-1}] = \mathfrak{g}^N = 0$, so $\mathfrak{g}^{N-1} \subset Z(\mathfrak{g})$.

(b) $\overline{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, i.e., $\overline{\mathfrak{g}}^n = 0$ for some n which implies $\mathfrak{g}^n \subset Z(\mathfrak{g})$, but then $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subset [\mathfrak{g}, Z(\mathfrak{g})] = 0$.

□

4.3 Engel's Characterization of Nilpotent Lie Algebras

Theorem 4.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent iff for each $a \in \mathfrak{g}$, $(ad\ a)^n = 0$ for some $n > 0$. One may always take $n = dim\mathfrak{g}$.*

proof If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}^{n+1} = 0$ for some n. In particular, $(\text{ad a})^n b = 0$ for all $a, b \in \mathfrak{g}$ sinc this is a length $(n+1)$ commutator. For the converse: The adjoint representation gives an injective homomorphism $\mathfrak{g}/Z(\mathfrak{g}) \hookrightarrow gl_{\mathfrak{g}}$ and by assumption the image consists of nilpotent operators. So by Engel's Theorem (from last lecture), $\mathfrak{g}/Z(\mathfrak{g})$ consists of strictly upper triangular matrices in the same basis. Therefore $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent and hence $\mathfrak{g}$ is nilpotent as well.

4.4 How to Classify 2-Step Nilpotent Lie Algebras

Let $\mathfrak{g}$ be n-dimensional and nilpotent with $Z(\mathfrak{g}) \neq 0$ so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent of dimension $n_1 < n$.

Definition 4.3.

- $\mathfrak{g}$ is *1-step nilpotent* if it is abelian
- $\mathfrak{g}$ is *2-step nilpotent* if $\mathfrak{g}/Z(\mathfrak{g})$ is abelian
- $\mathfrak{g}$ is *k-step nilpotent* if $\mathfrak{g}/Z(\mathfrak{g})$ is $(k-1)$-step nilpotent

Let $\mathfrak{g}$ be 2-step nilpotent so $V = \mathfrak{g}/Z(\mathfrak{g})$ is abelian. Consider the bilinear form

$$\text{B: V} \times V \to Z(\mathfrak{g})$$
$$\text{(a,b)} \mapsto [\tilde{a}, \tilde{b}]$$

where $\tilde{a}$ and $\tilde{b}$ are preimages of a, b under $\mathfrak{g} \to V$ (B is an *alternating form*, i.e., $B(x, x) = 0$ for all x).

Exercise 4.3. Show that 2-step nilpotent Lie algebras are classified by such nondegenerate alternating bilinear forms.

Proof. Suppose $\mathfrak{g}$ is a 2-step Lie algebra so $\mathfrak{g}/Z(\mathfrak{g})$ is abelian. Let $W = Z(\mathfrak{g})$ and $V \cong \mathfrak{g}/Z(\mathfrak{g})$. We check that the form $\phi : (v_1, v_2) \to [v_1, v_2]$ is nondegenerate and alternating. It is clearly alternating by the definition of the bracket, and nondegenerate since if $[v, v] = 0$ then $v \in Z(\mathfrak{g}) \Rightarrow v = 0$. For the other direction, given a triple (v, w, ϕ) such that $\phi : v \times v \to w$, consider $\mathfrak{g} = v \oplus w$. Then 2-step nilpotent Lie algebras with bracket $[v+w, v^{'}+w^{'}] = \phi(v, v^{'})$. This is the case because the bracket is alternating by the definition of ϕ, the bracket satisfies the Jacobi identity (see the next paragraph), and since the bracket is nondegenerate, $V \cong \mathfrak{g}/Z(\mathfrak{g})$. To check that the bracket satisfies the Jacobi identity, check that

$$\begin{aligned}
[v_1, [v_2, v_3]] + [v_2, [v_3, v_1] + [v_3, [v_1, v_2]] &= \phi(v_1, \phi(v_2, v_3)) + \phi(v_2, \phi(v_3, v_1)) + \phi(v_3, \phi(v_1, v_2)) \\
&= \phi(v_1, 0) + \phi(v_2, 0) + \phi(v_3, 0) = 0
\end{aligned}$$

We must show these maps are isomorphisms. Let $\alpha : \mathfrak{g} \to (v, w, \phi)$ and $\beta : (v, w, \phi) \to \mathfrak{g}$. We check that $\alpha\beta \cong 1_{(v,w,\phi)}$ and $\beta\alpha \cong 1_{\mathfrak{g}}$. We've seen that β sends a triple to the Lie algebra $v \oplus w$ where w is the center of the form $[v + w, v^{'} + w^{'}] = \phi(v, v^{'}) \in w$ which gets mapped to $(v \oplus w, w, \phi) \cong (v, w, \phi)$ The other direction $\mathfrak{g} \to (\mathfrak{g}/Z(\mathfrak{g}), Z(\mathfrak{g})) \to \mathfrak{g}/(Z(\mathfrak{g}) \oplus Z(\mathfrak{g}))$ giving a bijection. □

You can show that the problem of classifying all nilpotent algebras is equivalent to problems that are known to be impossible. However, you can classify things in some special circumstances.

Exercise 4.4. Show that if $Z(\mathfrak{g}) = \mathbb{F}c$ and $\mathfrak{g}$ is 2-step nilpotent, then $\mathfrak{g}$ is isomorphic to $H_{2n+1} = (\mathbb{F}p_1 + \mathbb{F}p_2 + \ldots + \mathbb{F}p_n) + (\mathbb{F}q_1 + \mathbb{F}q_2 + \ldots + \mathbb{F}q_n) + \mathbb{F}c$ with $[p_i, q_j] = \delta_{ij}$, $[p_i, p_i] = 0$, $[q_i, q_j] = 0$, and $[c, H_{2k+1}] = 0$.

Proof. From the previous exercise there exists a skew-symmetric form B on $V := \mathfrak{g}/\mathbb{F}c$, $B(v_1, v_2) = [v_1, v_2]$ and just as above it is easy to check that it is nondegenerate. But for any non-degenerate skew-symmetric bilinear form B on V over any field $\mathbb{F}$ there exists a basis p_i, q_i such that $B(p_i, q_j) = \delta_{ij}$, $B(p_i, p_j) = 0$, and $B(q_i, p_j) = 0$. Indeed, pick arbitrary $p_1 \in V$ and a q_1 such that $B(p_1, q_1) = 1$, and let $V_1^{\perp}$ be the orthocomplement to $\mathbb{F}p_1 + \mathbb{F}q_1$ in V. Continue by induction on $\dim V$. Then look at the preimages of these p_i and q_i in the space $\mathfrak{g}$, and note that they satisfy the same commutation relations. This implies that $H = (\mathbb{F}p_1 + \mathbb{F}p_2 + \ldots + \mathbb{F}p_n) + (\mathbb{F}q_1 + \mathbb{F}q_2 + \ldots + \mathbb{F}q_n) + \mathbb{F}c$ with the desired commutation relations. □