Lecture 3 — Engel's Theorem

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Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and V a vector space over $\mathbb{F}$. A *representation* of $\mathfrak{g}$ in V is a homomorphism $\pi : \mathfrak{g} \to gl_V$. In other words, it is a linear map $a \to \pi(a)$ from $\mathfrak{g}$ to the space of linear operators on V such that $\pi([a,b]) = \pi(a)\pi(b) - \pi(b)\pi(a)$.

Example 3.1. *Trivial representation* of $\mathfrak{g}$ in V where $\pi(a) = 0$ for all a.

Example 3.2. *Adjoint representation* of $\mathfrak{g}$ in $\mathfrak{g}$: $a \to \mathbf{ad}a$.

Let's check that it is in fact a representation. We must show that

$$\mathbf{ad}[a,b] = (\mathbf{ad}a)(\mathbf{ad}b) - (\mathbf{ad}b)(\mathbf{ad}a).$$

Applying both sides to $c \in \mathfrak{g}$, we check

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]].$$

By skew-symmetry, this is just the Jacobi Identity.

Definition 3.2. The *center* of a Lie algebra $\mathfrak{g}$ is denoted $Z(\mathfrak{g}) = \{a \in \mathfrak{g} | [a, \mathfrak{g}] = 0\}$.

Clearly, $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Exercise 3.1. Show $Z(gl_n(\mathbb{F})) = \mathbb{F}I_n$, $Z(sl_n(\mathbb{F})) = 0$ if *char* $\mathbb{F} \nmid n$.

Proof. This is clear when $n = 1$ so assume otherwise. Suppose $A = (a_{ij}) \in Z(gl_n(\mathbb{F}))$ and there exists $a_{ij} \neq 0$ with $i \neq j$. Let $B = (b_{ij})$ and consider $AB = C = (c_{ij})$ and $BA = C' = (c'_{ij})$. We wish to show $C \neq C'$ for some B. We have

$$c_{i1} = \sum_{k=1}^{n} a_{ik} b_{k1}$$

and

$$c'_{i1} = \sum_{k=1}^{n} b_{ik} a_{k1}.$$

Define B by $b_{j1} = 1$ and $b_{k1} = 0$ for all $k \neq j$. We then have $c_{i1} = a_{ij} \neq 0$ and that the i^{th} row of B only has the restriction $b_{i1} = 0$. We choose the remaining entries of the row so that $c'_{i1} \neq c_{i1} = a_{ij}$. Therefore $a_{ij} = 0$ for all $i \neq j$. We observe that these restrictions still allow B to have trace zero and determinant non-zero.

Now suppose without loss of generality that $a_{11} \neq a_{22}$. Consider

$$c_{12} = \sum_{k=1}^{n} a_{1k} b_{k2}$$

and let $b_{12} = 1$ and $b_{i2} = 0$ for all $i \neq 1$. These restrictions on B still allow $B \in sl_n(\mathbb{F}) \subset gl_n(\mathbb{F})$. Clearly $\mathbb{F}I_n \subset Z(gl_n(\mathbb{F}))$, so by above $\mathbb{F}I_n = Z(gl_n(\mathbb{F}))$. As well, since we allowed $B \in sl_n(\mathbb{F})$ and $tr(A) = na_{11}$, if *char* $\mathbb{F} \nmid n$, then $Z(sl_n(\mathbb{F})) = 0$. □

Proposition 3.1. *The adjoint representation defines an embedding of $\mathfrak{g}/Z(\mathfrak{g})$ in $gl_{\mathfrak{g}}$.*

Proof. $\mathbf{ad} : \mathfrak{g} \to gl_{\mathfrak{g}}$ is a homomorphism; Ker $\mathbf{ad} = Z(\mathfrak{g})$. Hence $\mathbf{ad}$ induces an embedding $\mathfrak{g}/Z(\mathfrak{g}) \to gl_n$ since $\mathfrak{g}/$ Ker $\varphi \cong$ Im φ. $\square$

Theorem 3.2. *Ado's Theorem*

Any finite dimensional Lie algebra embeds in $gl_n(\mathbb{F})$ for some n. [Presented without proof.]

Remark 1. Proposition 3.1 proves Ado's Theorem in the case $Z(\mathfrak{g}) = 0$.

Exercise 3.2. Let dim $\mathfrak{g} < \infty$. Show dim $Z(\mathfrak{g}) \neq$ dim $\mathfrak{g} - 1$.

Proof. Suppose $\dim Z(\mathfrak{g}) = \dim \mathfrak{g} - 1$ and pick any non-zero $x \in \mathfrak{g} \setminus Z(\mathfrak{g})$. Clearly, x commutes with $Z(\mathfrak{g})$ and with αx which implies $x \in Z(\mathfrak{g})$. This is a contradiction and therefore $\dim Z(\mathfrak{g}) \neq \dim \mathfrak{g} - 1$. $\square$

Definition 3.3. We define $Heis_{2n+1}$ to be the Lie algebra with basis $\{p_i, q_i, c\}$ where $[p_i, q_i] = c = -[q_i, p_i], 1 \leq i \leq n$, and all other bracketed pairs are 0.

Exercise 3.3. Classify all finite dimensional Lie algebras for which dim $Z(\mathfrak{g}) =$ dim $\mathfrak{g} - 2$. Let $\dim \mathfrak{g} = n$ and show either $\mathfrak{g} \cong Ab_{n-3} \oplus$ Heis$_3$ or $\mathfrak{g} \cong Ab_{n-2} \oplus \mathfrak{h}$ where $\mathfrak{h}$ is the two-dimensional non-abelian Lie algebra.

Proof. Suppose $\dim Z(\mathfrak{g}) = \dim \mathfrak{g} - 2$, then $\mathfrak{g}/Z(\mathfrak{g})$ may be generated by two elements, and consider their preimages, say p and q. Let $[p, q] = c \neq 0$ (else $p, q \in Z(\mathfrak{g})$. Suppose $c \in Z(\mathfrak{g})$, then in this case $\mathfrak{g} \cong Ab_{n-2} \oplus Heis_3$. Assume otherwise, that $c = z + a_p p + a_q q$ (without loss of generality, assume $a_p \neq 0$). We have $[c, q] = a_p[p, q] = a_p c$. Let $q' = \frac{q}{a_p}$, then c, q' are linearly independent and $[c, q'] = c$ which shows $\mathfrak{g} \cong Ab_{n-2} \oplus$ two-dimensional non-abelian Lie algebra. $\square$

Constructions of representations from given ones.

Definition 3.4. Representation from direct sum

Given representations π_1, π_2 of $\mathfrak{g}$ in V_i. We have $\pi_1 \oplus \pi_2$ of $\mathfrak{g}$ in $V_1 \oplus V_2 : (\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$.

Definition 3.5. Subrepresentation and factor representation

Given a representation of π of $\mathfrak{g}$ in V, if $U \subset V$ is a subspace invariant with respect to all operators $\pi(a), a \in \mathfrak{g}$, we have the subrepresentation π_U of $\mathfrak{g}$ in $U : a \to \pi(a)|_U$.

Moreover, the factor representation $\pi_{V/U}$ of $\mathfrak{g}$ in $V/U : a \to \pi(a)|_{V/U}$.

Definition 3.6. A linear operator A in a vector space V is called nilpotent if $A^N = 0$ for some positive integer N.

Exercise 3.4. Show if dim $V < \infty$, then A is nilpotent if and only if all eigenvalues of A are zero.

Proof. If λ is an eigenvalue of A, then λ^N is an eigenvalue of $A^N = 0$, and therefore $\lambda = 0$. Conversely, suppose all eigenvalues of A are zero, then the characteristic polynomial of A is t^n, and by Cayley-Hamilton $A^n = 0$. $\square$

Lemma 3.3. *Let A be a nilpotent operator in a vector space V, then*

(a) There exists a non-zero $v \in V$ such that $Av = 0$.

*(b) **ad** A is a nilpotent operator on gl_V.*

Proof. (a) Consider minimal $N > 0$ such that $A^N = 0$, then $A^{N-1} \neq 0$. Choose a non-zero vector $v \in A^{N-1}V \neq 0$. Then $Av = 0$.

Remark 2. **ad** $A = L_A - R_A$

$L_A(B) = AB$, $R_A(B) = BA$

$L_A R_B = R_B L_A$, due to the associativity of the product of operators. Hence

$$(\mathbf{ad}\ A)^M = \sum_{j=0}^{M} \binom{M}{j} L_A^j R_A^{M-j}.$$

(b) Now we have

$$\mathbf{ad}^{2N} B = \sum_{j=0}^{2N} \binom{2N}{j} A^j B A^{2N-j} = 0$$

since either $j \geq N$ or $2N - j \geq N$.

$\square$

Theorem 3.4. *Engel's Theorem Let V be a non-zero vector space and let $\mathfrak{g} \in gl_V$ be a finite dimensional subalgebra which consists of nilpotent operators. Then there exists a non-zero vector $v \in V$ such that $Av = 0$ for all $A \in \mathfrak{g}$.*

Proof. By induction on $\dim \mathfrak{g}$.

If $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \mathbb{F}a$ for $a \in gl_V$. By Lemma 3.3(a), Engel's Theorem holds.

We may assume $\dim \mathfrak{g} \geq 2$ and let $\mathfrak{h}$ be a maximal proper subalgebra of $\mathfrak{g}$. Since $\mathbb{F}a$ is always a subalgebra, we have that $\dim \mathfrak{h} \geq 1$. Consider the adjoint representation of $\mathfrak{g}$ (on itself) and consider its restriction to $\mathfrak{h}$, so we have $\mathbf{ad} : \mathfrak{h} \to gl_{\mathfrak{g}}$ is an invariant subspace for the representation (since $\mathfrak{h}$ is a subalgebra). Therefore, we may consider the factor representation in $\mathfrak{g}/\mathfrak{h}$. Then $\pi(\mathfrak{h}) \subset gl_{\mathfrak{g}/\mathfrak{h}}$ and $\dim \pi(\mathfrak{h}) \leq \dim \mathfrak{h} \leq \dim \mathfrak{g}$. Moreover, $\pi(\mathfrak{h})$ consists of nilpotent operators by Lemma 3.3(b).

We may apply the inductive assumption. We have there exists $\bar{a} \in \mathfrak{g}/\mathfrak{h}$, a non-zero vector such that $\pi(h)\bar{a} = 0$ for all $h \in \mathfrak{h}$. If $a \in \mathfrak{g}$ is an arbitrary preimage of $\bar{a}$, we get that $[\mathfrak{h}, a] \subset \mathfrak{h}$ and since $\bar{a} \neq 0$, $a \notin \mathfrak{h}$. Hence, $\mathfrak{h} \oplus \mathbb{F}a$ is a subalgebra of $\mathfrak{g}$. This subalgebra is larger than $\mathfrak{h}$, but $\mathfrak{h}$ was a maximal proper subalgebra of $\mathfrak{g}$, which implies $\mathfrak{h} \oplus \mathbb{F}a = \mathfrak{g}$.

By inductive assumption, there exists non-zero $v \in V$ such that $Av = 0$ for all $A \in \mathfrak{h}$. Let V_0 denote the space of all vectors $v \in V$ satisfying $Av = 0$. We claim that $aV_0 \subset V_0$; indeed: $V_0 = \{v \mid \mathfrak{h}v = 0\}$. So if $v \in V_0$, then we have $h(av) = [h, a] + ah(v) = 0 + 0 = 0$. By Lemma 3.3$(a)$ there exists a non-zero vector $v \in V_0$ "killed" by a. Therefore v is "killed" by $\mathfrak{h}$ and a, and hence $\mathfrak{g}$. $\square$

Remark 3. If we assume $\dim V < \infty$, then $\mathfrak{g}$ is finite dimensional since $\dim \mathfrak{g} \leq (\dim V)^2 < \infty$. Therefore Engel's Theorem holds if we only assume $\dim V < \infty$.

Corollary 3.5. *Let $\pi : \mathfrak{g} \to gl_V$ be a representation of a Lie algebra $\mathfrak{g}$ in a finite dimensional vector space V such that $\pi(a)$ is a nilpotent operator for all $a \in \mathfrak{g}$. Then there exists a basis of V in which all operators $\pi(a)$, $a \in \mathfrak{g}$ are strictly upper triangular matrices.*

Proof. Induction on $\dim V$.

By Engel's Theorem, there exists a non-zero vector e_1, such that $\pi(e_1) = 0$ for all $a \in \mathfrak{g}$. Since $\mathbb{F}e_1$ is an invariant subspace, we consider the factor representation of $\mathfrak{g}$ in $V/\mathbb{F}e_1$. Apply the inductive assumption to get the basis $\bar{e_2}, ..., \bar{e_n}$ of $V/\mathbb{F}e_1$, in which all matrices of $\pi|_{V/\mathbb{F}e_1}$ are strictly upper triangular. Take $e_2, ..., e_n$ preimages of $\bar{e_2}, ..., \bar{e_n}$. Then in the basis $e_1, ..., e_2$ of V, all matrices $\pi(\mathfrak{g})$ are strictly upper triangular. $\square$

Exercise 3.5. Construct in $sl_3(\mathbb{F})$ a two-dimensional subspace consisting of nilpotent matrices, which do not have a common eigenvector.

Proof. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then the subspace generated by A and B is two dimensional. A has eigenspace generated by $(1, 0, 0)^t$ and B has eigenspace generated by $(0, 0, 1)^t$. Therefore A and B have no common eigenvectors. Any linear combination of A, B say $\alpha A + \beta B$ has characteristic polynomial $-\lambda^3$ and therefore is nilpotent.

$\square$