18.745 Introduction to Lie Algebras	Dec 9, 2010
Lecture 25 —Dimensions and Characters of Semisimple Lie Algebras	
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Let $\mathfrak{g}$ be as in the last lecture - finite dimesional semisimple lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra, and $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta_+ \subset \Delta$, as before, a system of simple roots. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$, $\mathfrak{b} = \mathfrak{h}_+ + \mathfrak{n}_+$, with $\mathfrak{b}$ - a Borel subalgebra, and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}_+$. Let $(\cdot, \cdot)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, let $\rho = \frac{1}{2}\Sigma_{\alpha\in\Delta_+}\alpha$. Let $\{E_i, H_i, F_i\}$ be the Chevalley generators satisfying $H_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i,\alpha_i)}$, $E_i \in \mathfrak{g}_\alpha$, $F_i \in \mathfrak{g}_{-\alpha}$ and such that $< E_i, H_i, F_i >$ form the standard basis of $\mathrm{sl}_2(\mathbb{F})$. Define the subset $P_+ = \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \mathbb{Z}_+ \text{ for all } i = 1, \dots r\}$

Theorem 25.1. *(Cartan) The $\mathfrak{g}$-modules $\{L(\Lambda)\}_{\Lambda\in P_+}$ are, up to isomorphism, all irreducible finite-dimensional $\mathfrak{g}$-modules. (Recall from previous lectures that $L(\Lambda)$ is the irreducible heigest weight module with heighest weight λ.)*

Theorem 25.2. *(H.Weyl Dimension formula) If $\Lambda \in P_+$, then* $\mathrm{dim}L(\Lambda) = \prod\limits_{\alpha\in\Delta_+} \frac{(\Lambda+\rho,\alpha)}{(\rho,\alpha)}$

Example 25.1. $\mathfrak{g} = sl_2(\mathbb{F}) =< E, F, H >$. Then all Verma modules $M(\Lambda)$, where $\Lambda \in \mathfrak{h}^* = \mathbb{F}$ since $\mathfrak{h}^* = \mathbb{F}H$, have basis $F^J v_\Lambda, j \in \mathbb{Z}_+$. By the key sl_2 lemma, $M(\Lambda)$ is irreducible unless $\Lambda = \Lambda(H) \in \mathbb{Z}_+$ In the latter case (by the same lemma) $EF^{\Lambda+1}v_\Lambda = 0$, hence $F^{\Lambda+1}v_\Lambda$ is a singular vector. So $M(\Lambda)$ is not irreducible. But $L(\Lambda) = M(\Lambda)/\mathcal{U}(sl_2(\mathbb{F})F^{\Lambda+1}v_\Lambda$ is irreducible since $F^j v_\Lambda$, $0 \le j \le m = \Lambda(H)$ are independent. So by fundamental sl_2 lemma, $\dim(L(\Lambda)) = m + 1$

Proof of Theorem 25.1. Recall that $\sigma_i =< E_i, F_i, G_i >\simeq sl_2(F)$ and $v_\Lambda \in L(\Lambda)$ satisfies $E_i(v_\Lambda) = 0, H_i(v_\Lambda) = \Lambda(H_i)v_\Lambda$. Hence by fundamental sl_2 lemma, $\Lambda(H_i) \in Z_+$. So $\mathrm{dim}L(\Lambda) < \infty \Rightarrow \Lambda \in P_+$. Coversely, if $\Lambda \in P_+$, then by Theorem 2, $\dim(\Lambda) < \infty$.

□

Lemma 25.1. *Recall $\rho = \frac{1}{2} \sum\limits_{\alpha\in\Delta_+} \alpha$. Then $\rho(H_i) = 1$*

Proof. Consider the reflection from the Weyl group corresponding to α_i:

$$r_{\alpha_i}\rho = r_{\alpha_i}(\frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha\in\Delta\setminus\{\alpha_i\}}\alpha) = \rho - \alpha_i$$

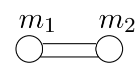
But $r_{\alpha_i}(\lambda) = \lambda - \lambda(H_i)\alpha_i$ hence $\rho(H_i) = 1$. □

Example 25.2. $\mathfrak{g} = \mathrm{sl}_2$, Then $\Lambda(H) = m$ means that $\Lambda = m\rho$. So

$$\mathrm{dim}L(\Lambda) = \frac{(m+1)(\alpha,\alpha)}{(\alpha,\alpha)} = m + 1$$

Example 25.3. $\mathfrak{g} = \mathrm{sl}_3$. We have $\Pi = \{\alpha_1, \alpha_2\}$, $(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\rho = \alpha_1 + \alpha_2$, $\rho(\alpha_i) = 1$, $\Lambda = m_1\Lambda_1 + m_2\Lambda_2$, where $(\Lambda_i, \alpha_j) = \delta_{ij}$. By Cartan's theorem, $\mathrm{dim}L(\Lambda) < \infty$ iff $m_1, m_2 \in \mathbb{Z}_+$. We compute $(\Lambda+\rho,\alpha_1)=m_1+1$, $(\Lambda+\rho,\alpha_2)=m_2+1$, and $(\Lambda+\rho,\alpha_1+\alpha_2)=m_1+m_2+1$, so we have

$$\dim L(\Lambda)=\frac{(m_1+1)(m_2+1)(m_1+m_2+1)}{2}$$

In general, we may write $\Lambda=\Sigma_i k_i\Lambda_i$, where $\Lambda_i(H_j)=\delta_{ij}$. Then $\dim L(\Lambda)<\infty$ iff $k_i\in\mathbb{Z}_+$. These k_i are called *labels* of the heighest weight. They are depicted on the Dynkin diagram: 

We'll deduce Weyl's Dimensional formula from the Weyl character formula

Definition 25.1. Let M be a $\mathfrak{g}$-module which is $\mathfrak{h}$-diagonalizable, let $M=\bigoplus\limits_{\lambda\in\mathfrak{h}^*}M_\lambda$, then

$$ch(M)=\sum_\lambda(\dim M_\lambda)e^\lambda$$

Here $e^\lambda e^\mu=e^{\lambda+\mu}, e^0=1$

Theorem 25.3. *(Weyl Character Formula) Let* $R=\prod\limits_{\alpha\in\Delta_+}(1-e^{-\alpha})$. *if* $\Lambda\in P_+$ *then*

$$e^\rho R\, chL(\Lambda)=\sum_{w\in W}(\det w)e^{w(\Lambda+\rho)} \qquad (*)$$

We'll first derive Theorem2 from Theorem3, then prove Theorem3.
Given $\mu\in h^*$, consider the following linear map from linear comibnation of the e^λ to functions of t that $F_\mu(e^\lambda)=e^{t(\lambda,\mu)}$.Apply F_μ to both sides of (*), we have

$$\begin{aligned} e^{t(\rho,\rho)}\prod(1-e^{-t(\rho,\alpha)})F_\rho chL(\Lambda) &= \sum_{w\in W}(\det w)e^{t(w(\Lambda+\rho),w^{-1}(\rho))}\\ &= \sum_{w\in W}(\det w)e^{t(w(\Lambda+\rho),w(\rho))}\\ &= F_{\Lambda+\rho}\sum_{w\in W}(\det w)e^{w(\rho)}\end{aligned}$$

Now we can value the sum.Note that $L(0)$ is the trivial 1-dim $\mathfrak{g}$-module, hence $chL(0)=1$. Therefore (*) implies

$$e^\rho R=\sum_{w\in W}(\det w)e^{w(\rho)} \qquad (**)$$

The above equality becomes

$$e^{t(\rho,\rho)}(F_\rho ch(L(\Lambda)))=\frac{F_{\Lambda+\rho}e^\rho R}{\prod\limits_{\alpha\in\Delta_+}(1-e^{-t(\rho,\alpha)})}=\prod_{\alpha\in\Delta_+}\frac{(1-e^{-t(\Lambda+\rho,\alpha)})}{(1-e^{-t(\rho,\alpha)})}$$

As $t\to 0$, $e^{t(\rho,\rho)}\to 1$. hence

$$LHS=F_\rho ch(L(\Lambda))=\sum_\lambda \dim L(\Lambda)e^{t(\rho,\Lambda)}=\dim L(\Lambda)$$

And by L'Hospitals rule,

$$\lim_{t\to 0}RHS=\lim_{t\to 0}\prod_{\alpha\in\Delta_+}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}\frac{e^{-t(\lambda+\rho,\alpha)}}{e^{-t(\rho,\alpha)}}=RHS \text{ of Weyl dimension fomula}$$

Proof of the Weyl's character formula. □

Lemma 25.2. *If $\Lambda(H_i) \in \mathbb{Z}_+$, then $chL(\Lambda)$ is r_i-invariant.*

Proof. By the key sl_2 lemma, $E_i F_i^{\Lambda(H_i)+1} v_\Lambda = 0$, $E_j F_i^{\Lambda(H_i)+1} v_\Lambda = 0$ for $j \neq i$ since E_j and F_i commute. So $F_i^{\Lambda(H_i)+1} v_\Lambda$ is a singular vector of $L(\Lambda)$ and since $L(\Lambda)$ is irreducible, so it has no singular weights other than Λ, we conclude that $F_i^{\Lambda(H_i)+1} v_\Lambda = 0$.

But $L(\Lambda) = \mathcal{U}(\mathfrak{g}) v_\Lambda$, hence for $v \in L(\Lambda)$ we conclude that $F_i^N v = 0$ for $N \gg 0$ Also obviously $E_i^N v = 0$ for $N \gg 0$.

It follows that any $v \in L(\Lambda)$ lies in a sl_2-invariant finite dimensional subspace. Hence by Weyl's complete reducibility theorem $L(\Lambda)$ is a direct sum of irreducible sl_2-modules. So it suffices to prove that the character of a finite dimensional irreducible sl_2-modules is r_{α_i}-invariant. Note that

$$chL(m\rho) = e^{m\rho} + e^{(m-2)\rho} + \ldots + e^{-m\rho}$$

Since $r_{\alpha_i}(\rho) = \rho - r_{\alpha_i}$, the character is r_{α_i}-invariant. Hence the lemma holds for $L(\Lambda)$ as well. □

Lemma 25.3. $R\mathrm{ch}M(\Lambda) = e^\Lambda$

Proof. We know from last lecture that vectors $E_{-\beta_1}^{m_1} \ldots E_{-\beta_N}^{m_N}$ form a basis of $M(\Lambda)$. Hence

$$\begin{aligned} chM(\Lambda) &= \sum_{(m_1,\ldots m_N) \in \mathbb{Z}_+^N} e^{\Lambda - m_1\beta_1 \ldots - m_N\beta_N} \\ &= e^\Lambda \sum_{(m_1,\ldots m_N) \in \mathbb{Z}_+^N} e^{\Lambda - m_1\beta_1 \ldots - m_N\beta_N} \\ &= \frac{e^\Lambda}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})} \end{aligned} \tag{1}$$

by geometric progression. □

Lemma 25.4. $w(e^\rho R) = (\det w) e^\rho R$ *for any* $w \in W$

Proof. Since W is generated by r_{α_i}, it suffices to check $r_{\alpha_i}(e^\rho R) = -e^\rho R$. Indeed, we can rewrite R as

$$R = (1 - e^{\alpha_i}) \prod_{\alpha \in \Delta_+ \setminus \alpha_i} (1 - e^{-\alpha})$$

Note that $\prod_{\alpha \in \Delta_+ \setminus \alpha_i} (1 - e^{-\alpha})$ is r_{α_i}-invariant, so by Lemma 1

$$\begin{aligned} r_{\alpha_i}(e^\rho R) &= e^{\rho - \alpha_i}(1 - e^{\alpha_i}) \prod_{\alpha \in \Delta_+ \setminus \alpha_i} (1 - e^{-\alpha}) \\ &= e^\rho (e^{-\alpha_i} - 1) \prod_{\alpha \in \Delta_+ \setminus \alpha_i} (1 - e^{-\alpha}) \\ &= -e^\rho R \end{aligned} \tag{2}$$

as wanted. □

Lemma 25.5. *Let $\Lambda \in \mathfrak{h}^*_{\mathbb{R}}$ and let V be a highest weight module with highest weight Λ. Let $D(\Lambda) = \{\Lambda - \Sigma k_i\alpha_i, k_i \in \mathbb{Z}_+\}$. Then $chV = \Sigma_{\lambda\in B(\Lambda)} a_\lambda\ chL(\lambda)$, where $B(\Lambda) = \{\lambda \in D(\Lambda) | (\Lambda + \rho, \Lambda + \rho) = (\lambda + \rho, \lambda + \rho)$ and $a_\Lambda = 1, a_\lambda \in \mathbb{Z}_+\}$.*

Proof. Proof is by induction on $\dim V = \sum_{\lambda\in B(\Lambda)} \dim V_\lambda < \infty$ due to Theorem2(h) from last lecture that $|B(\Lambda)| < \infty$ is finite. If $\sum_{\lambda\in B(\Lambda)} = 1$ then Λ is the only singular weight hence by Theorem2(c) from last lecture that $V = L(\Lambda)$ so $chV = chL(\Lambda)$. If there another singular vector $v_\lambda, \lambda \neq \Lambda$, let $U = \mathcal{U}(\mathfrak{g})v_\lambda$ and consider the following exact sequence of $\mathfrak{g}$-modules

$$0 \to U \to V \to V/U \to 0$$

Then $ch(V) = ch(U) + ch(U/V)$, now we apply the induction assumption to each of the terms. □

Lemma 25.6. *In the assumptions of Lemma 5 and $V = L(\Lambda)$ is irreducible, we have $chV = \Sigma_{\lambda\in B(\lambda)} b_\lambda chM_\lambda$, where $b_\Lambda = 1$ and $b_\lambda \in Z$.*

Proof. By Lemma 5 we have for any $\mu \in B(\Lambda)$: $chM(\mu) = \sum_{\lambda\in B(\mu)} a_{\lambda,\mu} chL(\lambda)$. Let $B(\Lambda) = \{\Lambda = \lambda_1, \ldots, \lambda_r\}$. Order them in such a way that $\lambda_i - \lambda_j \notin \{\sum_i k_i\alpha_i | k_i \in \mathbb{Z}_+\}$ if $i > j$. We get a system of equations $chM_{\lambda_j} = \sum_i a_{ij} chL(\lambda_i)$, where $a_{ii} = 1, a_{ij} = 0$ for $i > j$. So the matrix a_{ij} of this system is upper triangular matrix of integers with 1's on the diagonal and so its inverse, which expresses ch$L(\Lambda)$'s in terms of ch$M(\mu)$'s for $\mu \in B(\Lambda)$ is a matrix of integers with ones on the diagonal as well, and we are done. □

Proof. Proof of Theorem 3. With Lemma 6, $chL(\Lambda) = \sum_{\lambda\in B(\lambda)} b_\lambda \mathrm{ch}M(\lambda)$, where $b_\Lambda = 1$. Multiply both sides by $e^\rho R$ we get from Lemma 3 that

$$e^\rho R chL(\Lambda) = \sum_{\lambda\in B(\Lambda)} b_\lambda e^{\lambda+\rho} \qquad (!)$$

By Lemma 2 $L(\Lambda)$ is W-invariant, hence by Lemma 4, $e^\rho R chL(\Lambda)$ is W-anti-invariant (i.e. multiplied by the determinant). Hence the left hand side of the equation is anti-invariant, and therefore so is the right hand side. Hence using any simple transitivity of W on weyl chambers we can rewrite (!) as follows:

$$e^\rho R chL(\Lambda) = \sum_{w\in W} (\det w) e^{w(\Lambda+\rho)} + \sum_{\lambda\in B(\Lambda)\setminus\{\Lambda\}, \lambda+\rho\in P_+} b_\lambda \sum_{w\in W} (\det w) e^{w(\lambda+\rho)}$$

So it remains to show that the second term in this sum is 0, i.e. we need to prove that $\{\lambda \in B(\Lambda) \setminus \{\Lambda\}$ s.t. $\lambda + \rho \in P_+\} = \varnothing$. Note that $\lambda \in B(\Lambda)$ is of the form $\Lambda - \sum_i k_i\alpha_i, k_i \in Z_+$. Since $B(\Lambda) \subset D(\Lambda)$ and also $(\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho)$. Hence

$$\begin{aligned} 0 &= (\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho) \\ &= (\Lambda - \lambda, \lambda + \Lambda + 2\rho) \\ &= (\sum_i k_i\alpha_i, \Lambda) + (\sum_i k_i\alpha_i, \lambda) + 2(\sum_i k_i\alpha_i, \rho) \end{aligned}$$

since $(\Lambda, \alpha_i) = \frac{2\Lambda(H_i)}{(\alpha_i,\alpha_i)} \geq 0$ and similarly $(\lambda + \rho, \alpha_i) \geq 0$, and $(\rho, \alpha) = (\alpha_i, \alpha_i)/2 > 0$ This gives a contradiction so we complete the proof. $\square$