18.745 Introduction to Lie Algebras	December 2, 2010

Lecture 23 — Decomposition of Semisimple Lie Algebras

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Notation 23.1. First, we recall some facts from lecture 22. Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module.

- $Z^1(g, v) = \{f : \mathfrak{g} \mapsto V | f([a, b]) = af(b) - bf(a)\}$ is the space of 1-cocycles.
- $Z^1(\mathfrak{g}, v)$ contains the subspace $B^1(\mathfrak{g}, V) = \{f_v | f_v(a) = av\}$ of trivial 1-cocycles.
- $H^1(\mathfrak{g}, v) = Z^1(\mathfrak{g}, V)/B^1(\mathfrak{g}, V)$ is the first cohomology.
- $\Omega = \sum_j a_j b_j$ and is called the Casamir operator.

From now on, $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. We will prove that $H^1(\mathfrak{g}, V) = 0$ for any $\mathfrak{g}$-module V. This will be used to prove the Weyl's Complete Reducibility Theorem and Levi's Theorem.

The following exercise follows from the definitions and will be used to prove $H^1(\mathfrak{g}, V)$ vanishes.

Exercise 23.1. $H^1(\mathfrak{g}, V_1 \oplus V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2)$, where the V_i are $\mathfrak{g}$-modules.

Proof. [Solution] First we show that $Z(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$. It is clear that $Z(\mathfrak{g}, V_1 \oplus V_2) \supset Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$. Furthermore, every 1-cocycle $\varphi \in Z(\mathfrak{g}, V_1 \oplus V_2)$ can be decomposed as $\pi_1 \circ \varphi \oplus \pi_2 \circ \varphi \in Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$.

It is also clear that $B(\mathfrak{g}, V_1 \oplus V_2) = B(\mathfrak{g}, V_1) \oplus B(\mathfrak{g}, V_2)$ since $\varphi_{v_1 \oplus v_2} = \varphi_{v_1} \oplus \varphi_{v_2}$.

Therefore $H^1(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1 \oplus V_2)/B(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)/B(\mathfrak{g}, V_1) \oplus B(\mathfrak{g}, V_2) = Z(\mathfrak{g}, V_1)/B(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)/B(\mathfrak{g}, V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2)$. □

We will also use the following lemma from lecture 22 and a corollary.

Lemma 23.1. *If $\mathfrak{g}$ is a Lie algebra with an invariant non-degenerate bilinear form $(.,.)$, a_i, b_i are dual basis of $\mathfrak{g}$ (i.e $(a_i, b_j) = \delta_{ij}$) and $f \in Z^1(\mathfrak{g}, V)$:*

$$a \sum_j a_j f(b_j) = \Omega(f(a)), \forall a \in \mathfrak{g}.$$

Corollary 23.2. *The Casamir operator commutes with the action of $\mathfrak{g}$ on any $\mathfrak{g}$-module V i.e. $a(\Omega(v)) = \Omega(av)$ for any $a \in \mathfrak{g}, v \in V$.*

Proof. Take $f = f_v$. The equation in the lemma becomes:

$$a(\Omega(v)) = a \sum_j a_j b_j(v) = \Omega(a(v))$$

□

Theorem 23.3 (Vanishing Theorem). *If V is a finite dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, V) = 0$.*

Proof. By Corollary 23.2, Ω commutes with the action of $\mathfrak{g}$ on V. This means the generalized eigenspace decomposition for Ω acting on $V = \oplus_{\lambda \in \mathbb{F}} V_\lambda$ is $\mathfrak{g}$-invariant i.e. all the subspaces V_λ are $\mathfrak{g}$-invariant. Let $V' = \oplus_{\lambda \neq 0} V_\lambda$, so that $V = V_0 \oplus V'$. As both V_0 and V' are $\mathfrak{g}$-invariant, they both are $\mathfrak{g}$-submodules of V.

By Exercise 23.1 $H^1(\mathfrak{g}, V) = H^1(\mathfrak{g}, V_0) \oplus H^1(\mathfrak{g}, V')$. We will prove both of these terms equal 0. First, consider $H^1(\mathfrak{g}, V_0)$.

Let $f : \mathfrak{g} \mapsto V_0$ be a 1-cocycle.

Let $\mathfrak{g}_0 = \{a \in \mathfrak{g} | aV = 0\}$. This is the kernel of the representation $\mathfrak{g} \mapsto EndV$ and therefore an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, the ideal $\mathfrak{g}_0$ is also semisimple. Since $\mathfrak{g}_0$ is semisimple, $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$. For $a, b \in \mathfrak{g}_0$, $f[a, b]) = af(b) - bf(a) = 0 - 0 = 0$. $f[\mathfrak{g}_0, \mathfrak{g}_0] = 0$ and thus $f[\mathfrak{g}_0] = 0$. This means that $\bar{f} : \bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}_0 \mapsto V$ is a well-defined map.

Without loss of generality we may assume that $\mathfrak{g} = \bar{\mathfrak{g}}$, for any 1-cocycle of $\mathfrak{g}$ induces one on $\bar{\mathfrak{g}}$ and vice versa.

We now specify an invariant bilinear form on $\mathfrak{g}$. Take for $(.,.)$ on $\mathfrak{g}$ the trace form of $\mathfrak{g}$ on V_0: $(a, b)_V = tr_{V_0} ab$. By Cartan's criterion, this trace form is non-degenerate as $\bar{\mathfrak{g}}$ is a semisimple Lie algebra by assumption and the representation $\bar{\mathfrak{g}} \mapsto V_0$ is faithful by the construction of $\bar{\mathfrak{g}}$.

$$tr_V(\Omega) = \sum_i tr_V a_i b_i = \sum_i (a_i, b_i)_V = dim\mathfrak{g},$$

where a_i, b_i are dual basis.

$\Omega|_{V_0}$ is nilpotent and therefore $tr_V(\Omega) = 0$. This implies $\mathfrak{g} = 0$ which implies $H^1(\mathfrak{g}, V_0) = 0$.

Now to prove that $H^1(\mathfrak{g}, V') = 0$,

For any 1-cocycle $f : \mathfrak{g} \mapsto V'$, by Lemma 23.1, $a(m) = \Omega(f(a))$ where $m = \sum_i a_i(f(b_i)) \in V'$. But $\Omega|_{V'}$ is an invertible operator and $\Omega a = a\Omega$, so that $f(a) = \Omega^{-1}am = a\Omega^{-1}m$. $f = f_{\Omega^{-1}m}$ is a trivial 1-cocycle and $H^1(\mathfrak{g}, V') = 0$. ☐

Remark 23.1. A couple notes on the proof. When V is finite dimensional, $V = V_0 \oplus V'$ over any field $\mathbb{F}$ and operator Ω. Where $\Omega|_{V_0}$ is nilpotent and $\Omega|V'$ is invertible. We do not need to assume that $\mathbb{F}$ is algebraically closed.

The assumption of semisimplicity was used to prove a non-degenerate invariant bilinear form exists. Does the converse hold, that if $H^1(V, \mathfrak{g}) = 0$ then $\mathfrak{g}$ is semisimple?

With the Vanishing Theorem one can prove these theorems about semisimple Lie algebras.

Theorem 23.4 (Weyl Complete Reducibility Theorem). *If $char\mathbb{F} = 0$ and V is a finite dimensional module over a semisimple Lie algebra $\mathfrak{g}$, then for any submodule U of V there exists a complementary submodule U' so that $V = U \oplus U'$.*

Note that since V is finite dimensional it follow from this theorem that V is isomorphic to the direct sum of irreducible $\mathfrak{g}$-modules.

Theorem 23.5 (Levi's Theorem)**.** *If $\mathfrak{g}$ is a finite dimensional Lie algebra over a field $\mathbb{F}$ of characteristic* 0*, and $R(\mathfrak{g})$ is the radical then $\mathfrak{g}$ contains a subalgebra s complementary to $R(\mathfrak{g})$ Furthermore, $\mathfrak{g} = s \ltimes R(\mathfrak{g})$.*

Definition 23.1. s is called a *Levi factor.*

Theorem 23.6 (Mal'cev Thceorem)**.** *All Levi factors s in $\mathfrak{g}$ are conjugate to each other by an automorphsim of $\mathfrak{g}$.*

Remark 23.2. The proofs use projection operators in a vector space V. A projector P of a subspace U of V is an endomorphism of V such that: $P(V) \subset U$ and $P(u) = u$ for $u \in U$.

Note that if P_0 is a projection operator from V to U, then any other projector of V to U has the form $P = P_0 + A$ where $A(V) \subset U, A(u) = 0$.

Proof: Weyl Complete Reducibility Theorem. Consider the space End V with the following $\mathfrak{g}$-module structure: $a(A) = \pi(a)A - A\pi(a)$. where $a \in \mathfrak{g}, A \in$ End V $\pi : \mathfrak{g} \mapsto$ End V is the representation of $\mathfrak{g}$ on V. This satisfies the properties of a $\mathfrak{g}$-module by the Jacobi identity.

Fix a projector $P_0 : V \mapsto U$ and consider the corresponding trivial 1-cocycle. $f_{P_0} = a(P_0) = \pi(a)P_0 - P_0\pi(a)$. Let $M \subset EndV$ be the following subspace: $M = \{A|A(V) \subset U, A(U) = 0\}$. Let $A \in M$ then

$(\pi(a)A - A\pi(a))v = \pi(a)u - Av' \in U.$

$(\pi(a)A - A\pi(a))u = 0 - Au' = 0.$

We know $f_{P_0} : \mathfrak{g} \mapsto EndV$, but in fact $f_{P_0} : \mathfrak{g} \mapsto M$ since

$(\pi(a)P_0 - P_0\pi(a))v = \pi(a)u - P_0v' \in U$

$(\pi(a)P_0 - P_0\pi(a))u = \pi(a)u - \pi(a)u = 0$

Therefore, $f_{P_0} \in Z^1(\mathfrak{g}, M)$.

By the Vanishing Theorem, $H^1(\mathfrak{g}, M) = 0$, so f_{P_0} is a trivial 1-cocycle and$f_{P_0}(a) = A\pi(a) - \pi(a)A$ for some $A \in M$.Rearranging this gives: $(\pi(a))(P_0 - A) = (P_0 - A)(\pi(a))$ In other words, $P_0 - A$ is a new projector P of V with $a(P_0 - A) = 0$

Letting $U' =$ ker P we get $V = U \oplus U'$ where U' is $\mathfrak{g}$-invariant, since the projector P commutes with $\mathfrak{g}$.

U and U' are our complementary g-submodules. □

The proof of Levi's Theorem follows the same method.

Proof: Levi's Theorem. We prove it by induction on $dim\mathfrak{g}$. The case $dim\mathfrak{g} = 1$ is obvious.

Case 1 $R(\mathfrak{g})$ is not abelian. Consider the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g}/[R(\mathfrak{g}), R(\mathfrak{g})]$, where $dim\bar{\mathfrak{g}} < dim\mathfrak{g}$ and we apply the inductive assumption. $\bar{\mathfrak{g}} = \bar{s} \ltimes R(\bar{\mathfrak{g}})$. Let $\mathfrak{g}_1$ be the premiage of $\bar{s}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_1 + R(\mathfrak{g})$ with $dim\mathfrak{g}_1 < dim\mathfrak{g}$. By the inductive assumption, $\mathfrak{g}_1 = s \ltimes R(\mathfrak{g}_1)$. and $\mathfrak{g} = s \ltimes (R(\mathfrak{g}_1) + R(\mathfrak{g}))$.

Exercise 23.2. $R(\mathfrak{g}_1) + R(\mathfrak{g})$ is a solvable ideal of $\mathfrak{g}$, since $R(\mathfrak{g})$ is a maximal solvable ideal of $\mathfrak{g}$ so $R(\mathfrak{g}_1) \subset R(\mathfrak{g})$, show that g=$R(g) \ltimes s$.

Proof. [Solution]$R(\mathfrak{g}_1)+R(\mathfrak{g})$ is an ideal of $\mathfrak{g}$: Write an arbitrary element of $R(\mathfrak{g}_1)+R(\mathfrak{g})$ as $a+b$, where $a \in R(\mathfrak{g}_1)$ and $b \in R(\mathfrak{g})$. Write an arbitrary element of $\mathfrak{g}$ as $c+d$, where $c \in \mathfrak{g}_1$ and $d \in R(\mathfrak{g})$. Then $[a+b, c+d] = [a,c]+[a,d]+[b,c+d]$.

We have $[a,c] \in R(\mathfrak{g}_1)$ because $R(\mathfrak{g}_1)$ is an ideal of $\mathfrak{g}_1$, $[a,d],[b,c+d] \in R(\mathfrak{g})$ because $R(\mathfrak{g})$ is an ideal. Furthermore, $R(\mathfrak{g}_1)+R(\mathfrak{g})$ is solvable. Assume $R(\mathfrak{g}_1)^{(m)} = 0$, and $R(\mathfrak{g})^{(n)} = 0$,then $(R(\mathfrak{g}_1)+R(\mathfrak{g}))^{(m)} \subseteq R(\mathfrak{g}_1)^{(m)}+R(\mathfrak{g}) = R(\mathfrak{g})$. Hence, $(R(\mathfrak{g}_1)+R(\mathfrak{g}))^{(mn)} = 0$. Thus, we have a solvable ideal of $\mathfrak{g}$ which contains $R(\mathfrak{g})$, so we conclude that our ideal is in fact $R(\mathfrak{g})$, and $R(\mathfrak{g}_1) \subset R(\mathfrak{g})$. Hence we conclude

$$\mathfrak{g} = s \ltimes R(\mathfrak{g}).$$

□

Case 2 $R(\mathfrak{g})$ is abelian. Construct the following $\mathfrak{g}$-module structure on the space End $\mathfrak{g}$: $a(m) = (\mathbf{ad}\ a)m - m(\mathbf{ad}\ a), m \in End\mathfrak{g}$.

Let $\tilde{M} = \{m \in \text{End } \mathfrak{g} | m(\mathfrak{g}) \subset R(\mathfrak{g}), m(R(\mathfrak{g})) = 0\}$. Since $R(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, $\tilde{M}$ is a $\mathfrak{g}$-submodule of End $\mathfrak{g}$. Next let $\tilde{R} = \{\mathbf{ad}\ a | a \in R(\mathfrak{g})$, which is also a $\mathfrak{g}$-module.

Finally, put $M = \tilde{M}/\tilde{R}$. This is a $\mathfrak{g}$-module as it is the quotient of two g-modules.

Exercise 23.3. Show that M is an s-module and that $R(\mathfrak{g})$ acts on M trivally.

Proof. [Solution] $R(\mathfrak{g})$ acts trivially on M because $R(\mathfrak{g})$ is abelian. $[R(\mathfrak{g}), \tilde{M}] = 0$.Therefore, M is an s-module because we let $\tilde{s}$ be the preimage of s in $\tilde{M}$ and define $s(m) = \tilde{s}(m)$. As $R(\mathfrak{g})$ acts trivially, this is well-defined. □

Now fix a projector P_0 of $\mathfrak{g}$ on $R(\mathfrak{g})$.

Exercise 23.4. Prove that $f(a) = (\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a})$, where $\tilde{a}$ is the preimage of $a \in \mathfrak{g}$ under the canonical map $\mathfrak{g} \mapsto \mathfrak{g}/Rad(\mathfrak{g}) = s$, is a well-defined 1-cocycle of s in $M = \tilde{M}/\tilde{R}$.

Proof. [Solution] First we need to show that f is well defined on M. For any $r \in \tilde{R}$, $(\mathbf{ad}\ (\tilde{a}+r))P_0 - P_0(\mathbf{ad}\ (\tilde{a}+r)) = (\mathbf{ad}\ \tilde{a})P_0 + (\mathbf{ad}\ r)P_0 - P_0(\mathbf{ad}\ \tilde{a} + \mathbf{ad}\ r) = (\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a}) + 0 - (\mathbf{ad}\ r)$ This is because P_0 is the identity on $R(\mathfrak{g})$ and $R(\mathfrak{g})$ is abelian, so $\mathbf{ad}\ r(R(\mathfrak{g}) = 0$. f is well-defined on M.

To show that it is a 1-cocycle we compute the necessary identity: $f[a,b] = (\mathbf{ad}\ [\tilde{a},\tilde{b}])P_0 - P_0(\mathbf{ad}\ [\tilde{a},\tilde{b}])$. Here we use that the fact that the pullback map respects the Lie bracket.

$a(f(b)) - b(f(a)) = (\mathbf{ad}\ a)f(b) - f(b)(\mathbf{ad}\ a) - (\mathbf{ad}\ b)f(a) + f(a)(\mathbf{ad}\ b) = (\mathbf{ad}\ a)((\mathbf{ad}\ \tilde{b})P_0 - P_0(\mathbf{ad}\ \tilde{b})) - ((\mathbf{ad}\ \tilde{b})P_0 - P_0(\mathbf{ad}\ \tilde{b}))(\mathbf{ad}\ a) - (\mathbf{ad}\ b)((\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a})) + ((\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a}))(\mathbf{ad}\ b) = (\mathbf{ad}\ [\tilde{a},\tilde{b}])P_0 - P_0(\mathbf{ad}\ [\tilde{a},\tilde{b}]$ □

Applying the vanishing of $H^1(s, M)$, we conclude that for $a \in s$, $f(a) = f_m(a)$ for some $m \in M$. Let $\tilde{m}$ be a preimage of m in $\tilde{M}$ from the canonical map and put $P = P_0 - \tilde{m}$. P is a projector of $\mathfrak{g}$ on $R(\mathfrak{g})$ for which $P(s) = 0$.

It is immediate to check that $[P, \mathbf{ad}\ \mathfrak{g}] = [P, \mathbf{ad}\ R(\mathfrak{g})] \subset \mathbf{ad}\ R(\mathfrak{g})$. This gives two cases depending if P and $\mathfrak{g}$ commute or not:

Case 1 $[P, \mathbf{ad}\ \mathfrak{g}] = 0$

Exercise 23.5. Given that $[P, \mathbf{ad}\ \mathfrak{g}] = 0$, prove that ker P is an ideal of $\mathfrak{g}$ so $\mathfrak{g} = \ker P \ltimes R(\mathfrak{g})$.

Proof. [Solution] P is an endomorphism, so ker P is clearly a subalgebra. We need to show that $[\mathfrak{g}a] \in \ker P$. For $a \in \ker P$, $P([\mathfrak{g}, a]) = P(\mathbf{ad}\ \mathfrak{g}(a)) = \mathbf{ad}\ \mathfrak{g}P(a)$, since $[P, \mathbf{ad}\ \mathfrak{g}] = 0$, and $P(a) = 0$ as $a \in \ker P$, so $P([\mathfrak{g}, a]) = \mathbf{ad}\ \mathfrak{g}(0) = 0$ and $[\mathfrak{g}a] \in \ker P$.

$\mathbf{ad}\ : R(\mathfrak{g}) \mapsto \text{End} \ker P$ since ker P is an ideal of $\mathfrak{g} \supset R(\mathfrak{g})$. Therefore, we make $\ker P \ltimes R(\mathfrak{g})$. $\mathfrak{g} = \ker P \oplus R(\mathfrak{g})$ as vector spaces. The following is a Lie algebra homomorphism: $(k, r) \mapsto k + r \in \mathfrak{g}$ They clearly respect the opperations. The bracket is defined in $\ker P \ltimes R(\mathfrak{g})$ to agree with it in $\mathfrak{g}$. It is an isomorphism of vector spaces. Thus it is also a Lie algebra isomorphism. □

Case 2 $[P, \mathbf{ad}\ \mathfrak{g}] \neq 0$. Consider the subalgebra $g_1 = \{a \in g | [P, \mathbf{ad}\ a] = 0\}$. It is a subalgebra of $\mathfrak{g}$ such that $dim \mathfrak{g}_1 < dim \mathfrak{g}$. By the inductive assumption, $g_1 = s \ltimes Rad(g_1)$. Now we use that $[P, \mathbf{ad}\ \mathfrak{g}] \subset \mathbf{ad}\ R(\mathfrak{g})$. In other words, $[P, \mathbf{ad}\ b] = \mathbf{ad}\ r_b$, for any $b \in \mathfrak{g}$ there exists such $r_b \in R(\mathfrak{g})$. Since $(\mathbf{ad}\ r_b)P = 0$ as P is a projector with these properties) and $P\mathbf{ad}\ r_b = \mathbf{ad}\ r_b$, we get $[P, \mathbf{ad}\ r_b] = \mathbf{ad}\ r_b$. So $[P, \mathbf{ad}\ (b - r_b)] = 0$. This means $b - r_b \in \mathfrak{g}_1$. $\mathfrak{g} = \mathfrak{g}_1 + R(\mathfrak{g})$ as vector spaces. Apply the proof of Ex 23.2 we deduce that $\mathfrak{g} = s \ltimes R(g)$. □

This proves Levi's Theorem.