Let $V$ be a finite dimensional Euclidean vector space, i.e. a real vector space with a positive definite symmetric bilinear form $(\cdot,\cdot)$. Let $a \in V$ be a nonzero vector and denote by $r_a$ the orthogonal reflection relative to $a$, i.e. $r_a(a) = -a, r_a(v) = v$ if $(a,v) = 0$.

**Formula 21.1.** $r_a(v) = v - \frac{2(a,v)}{(a,a)}a$.

**Exercise 21.1.** Prove: (a) $r_a \in O_v(\mathbb{R})$, i.e. $(r_a u, r_a v) = (u,v), u,v \in V$. (b) $r_a = r_{-a}$ and $r_a^2 = 1$. (c) $\det r_a = -1$. (d) If $A \in O_v(\mathbb{R})$, then $Ar_aA^{-1} = r_A(a)$.

**Proof.** For (a), $(r_a u, r_a v) = (u - \frac{2(a,u)}{(a,a)}a, v - \frac{2(a,v)}{(a,a)}a) = (u,v) - \frac{2(a,u)(a,v)}{(a,a)}a + \frac{4(a,u)(a,v)(a,a)}{(a,a)^2} = (u,v)$. For (b), since $r_a(a) = -a$, we have $r_{-a}(-a) = a$. Also, for any $v$ perpendicular to $a$, it is also perpendicular to $-a$, thus $r_{-a}(v) = v$. Thus $r_a$ acts the same way as $r_{-a}$ on the entire space, so $r_a = r_{-a}$. Next, $r_a^2(a) = r_{-a}(a) = a$. Also, for $v$ perpendicular to $a$, $r_a^2(v) = r_a(v) = v$. Thus, $r_a$ acts the same way as 1 does, so $r_a^2 = 1$.

For (c), $r_a$ fixes the space (of dimension $\dim V-1$) perpendicular to $a$, thus it has 1 as an eigenvalue of multiplicity $\dim V - 1$. The other eigenvalue is $-1$, corresponding to eigenvector $a$. Thus $\det r_a$ equals product of all eigenvalues, which is $-1$.

For (d), notice that $Ar_aA^{-1}(A(a)) = Ar_a(a) = A(-a) = -A(a)$. Also, for any $v$ perpendicular to $A(a)$, $(A^{-1}(v), a) = (v, A(a)) = 0$ as $A \in O_v$. Thus $Ar_aA^{-1}(v) = A(A^{-1}(v)) = v$. Thus $Ar_aA^{-1}$ acts the same way as $r_A(a)$, so they are equal.

**Definition 21.1.** Let $(V, \Delta)$ be a root system. Let $W$ be the subgroup of $O_v(\mathbb{R})$, generated by all $r_a$, where $a \in \Delta$. The group $W$ is called the Weyl group of the root system $(V, \Delta)$ (and of the corresponding semisimple lie algebra $g$).

**Proposition 21.1.** (a) $w(\Delta) = \Delta$ for all $w \in W$. (b) $W$ is a finite subgroup of the group $O_v(\mathbb{R})$.

**Proof.** For (a), it suffice to show that $r_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \alpha \in \Delta$ if $\alpha, \beta \in \Delta$. First, $r_\alpha$ is nonsingular as it has determinant $-1$. Recall the string property of $(V, \Delta)$: $\{\beta - k\alpha| k \in \mathbb{Z}\} \cap (\Delta \cup \{0\}) = \{\beta - p\alpha, ..., \beta + q\alpha\}$, where $p, q \in \mathbb{Z}_+, p-q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Hence $p \geq \frac{2(\alpha,\beta)}{(\alpha,\alpha)}, q \geq -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. So if $(\alpha, \beta) \leq 0$, by the string property, we can add $\alpha$ to $\beta$ at least $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ times and if $(\alpha, \beta) \geq 0$ we can subtract $\alpha$ from $\beta$ at least $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ times, which exactly means that $\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \alpha \in \Delta \cup \{0\}$. But it can’t be 0 since $r_\alpha$ is nonsingular.

(b) is clear since $\Delta$ spans $V$, so if $w \in W$ fixes all elements of $\Delta$, it must be 1, so $W$ embeds in the group of permutations of the finite set $\Delta$ by (a). Therefore $W$ is finite.

**Remark 21.1.** (a) shows the string property of the root system implies that $\Delta$ is $W$-invariant. One can show converse is true: if we replace string property by $W$-invariance of $\Delta$ and that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$,
then we get an equivalent definition of a root system. One has to check only for the case \( \dim V = 2 \).

(see Serre)

Fix \( f \in V^* \) which doesn’t vanish on \( \Delta \), let \( \Delta_+ = \{ \alpha \in \Delta | f(\alpha) > 0 \} \) be the subset of positive roots and let \( \Pi = \{ \alpha_1, ..., \alpha_r \} \in \Delta_+ \) be the set of simple roots \( (r = \dim V) \). Then the reflections \( s_i = r_{\alpha_i} \) are called simple reflections.

**Theorem 21.2.** (a) \( \Delta_+ \{ \alpha_i \} \) is \( s_i \)-invariant. (b) if \( \alpha \in \Delta_+ \Pi \), then there exists \( i \) such that \( hts_i(\alpha) < h\alpha \) (ht \( \sum_i k_i \alpha_i = \sum_i k_i \)). (c) If \( \alpha \in \Delta_+ \Pi \), then there exists a sequence of simple reflections \( s_{i_1}, ..., s_{i_k} \) such that \( s_{i_1}, ..., s_{i_k}(\alpha) \in \Delta_+ \) for all \( 1 \leq j \leq k \). (d) The group \( W \) is generated by simple reflections.

**Proof.** (a) For a positive root \( \alpha, s_i(\alpha) = \alpha - n\alpha_i \), where \( n \) is an integer. If \( \alpha \neq \alpha_i \), then all coefficients in the decomposition of simple root remain positive, except possibly for the coefficient of \( \alpha_i \). But in a positive root \( \alpha \), all coefficients are nonnegative. Hence if \( \alpha \) is not simple, it must have positive coefficient in front of some simple roots other than \( \alpha_i \) (other positive integer multiples of \( \alpha_i \) are not in \( \Delta \) by definition of root system), thus \( s_i(\alpha) \) should remain positive.

(b) \( s_i(\alpha) = \alpha - 2\frac{(\alpha, \alpha)}{(\alpha, \alpha_i)} \alpha_i \). Now if \( hts_i(\alpha) \geq h\alpha \) for all \( i \), then \( (\alpha, \alpha) \leq 0 \) for all \( i \). But then \( (\alpha, \alpha) = \sum_i a_i(\alpha, \alpha_i) \leq 0 \). Since \( \alpha = \sum_i a_i \alpha_i, a_i \geq 0 \). So \( \alpha = 0 \), a contradiction.

(c) Just apply (b) finitely many times until we get a simple root.

(d) Denote by \( W' \) the subgroup of \( W \), generated by simple reflections. By (c), for any \( \alpha \in \Delta_+ \) there exists \( w \in W' \), such that \( w(\alpha) = \alpha_j \in \Pi \). Hence by Ex.21.1(d), \( r_\alpha = w^{-1}s_jw \), which lies in \( W' \). So \( W' \) contains all reflections \( r_\alpha \) with \( \alpha \in \Delta \). But \( r_{-\alpha} = r_\alpha \), so \( W' \) contains all reflections, hence \( W' = W \). \( \square \)

**Example 21.1.** \( \Delta_{Ar} = \{ \epsilon_i - \epsilon_j | 1 \leq i, j \leq r + 1, i \neq j \} \subset V = \{ \sum_{i=1}^{r+1} a_i \epsilon_i | \sum_i a_i = 0, a_i \in \mathbb{R} \} \subset \mathbb{R}^{r+1} = \bigoplus_{i=1}^{r+1} \mathbb{R} \epsilon_i \), with \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \).

We have \( r_{\epsilon_i - \epsilon_j}(\epsilon_s) = \epsilon_s - \frac{2(\epsilon_s, \epsilon_i - \epsilon_j)}{2}(\epsilon_i - \epsilon_j) = \begin{cases} \epsilon_s & \text{if } s \neq i, s \neq j \\ \epsilon_j & \text{if } s = i \\ \epsilon_i & \text{if } s = j \end{cases} \) So \( r_{\epsilon_i - \epsilon_j} \) is transposition of \( \epsilon_i, \epsilon_j \), so \( W_{Ar} = S_{r+1} \).

**Exercise 21.2.** Compute Weyl Group for root systems of type \( B_r, C_r, D_r \). In particular show that for \( B_r \) and \( C_r \) they are isomorphic, but not isomorphic to \( D_r \).

**Proof.** \( \Delta_{B_r} = \{ \pm \epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j \} \cup \{ \pm \epsilon_i \} \subset \mathbb{R}^r = \bigoplus_{i=1}^{r} \mathbb{R} \epsilon_i \), with \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \). It is easy to see that, as in the example, \( r_{\epsilon_i - \epsilon_j} \) switches \( \epsilon_i \) with \( \epsilon_j \), \( r_{\epsilon_i} \) switches the sign of \( \epsilon_i \) and the other \( r_\alpha \) are generated by the previous two. Thus the Weyl Group consisting of all elements that permute \( r \) elements as well as switch some of their signs. So it is the semidirect product group \( \mathbb{Z}_2^r \rtimes S_r \). The root system for \( C_r \) is just that for \( B_r \) with \( \pm \epsilon_i \) replaced by \( \pm 2 \epsilon_i \). So of course the reflections are exactly the same, so Weyl Groups for \( B_r \) and \( C_r \) are the same.

\( \Delta_{D_r} = \{ \pm \epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j \} \subset \mathbb{R}^r = \bigoplus_{i=1}^{r} \mathbb{R} \epsilon_i \), with \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \). Now \( r_{\epsilon_i - \epsilon_j} \) acts as before, and \( r_{\epsilon_i + \epsilon_j} \) switches \( \epsilon_i \) to \( -\epsilon_j \) and \( \epsilon_j \) to \( -\epsilon_i \). Thus Weyl Group is the group of all elements that permute \( r \) elements as well as switching an even number of their signs. Thus it is \( \mathbb{Z}_2^{r-1} \rtimes S_r \), and it is not isomorphic to that of \( B_r \) and \( C_r \). \( \square \)
Definition 21.2. Consider the open (in usual topology) set \( V - \cup_{\alpha \in \Delta} T_\alpha \) in \( V \) (\( T_\alpha \) is the hyperplane perpendicular to \( \alpha \)). The connected components of the set are called open chambers, the closures are called closed chambers. \( C = \{ v \in V | (\alpha_i, v) > 0, \alpha_i \in \Pi \} \) is called the fundamental chamber; \( \widehat{C} = \{ v | (\alpha_i, v) \geq 0, \alpha_i \in \Pi \} \) is called closed fundamental chamber.

Exercise 21.3. Show that the open fundamental chamber is a chamber.

Proof. We first show that \( C \) is connected. Given any \( u, v \in C \), we have \((\alpha_i, u) > 0 \) for all \( \alpha_i \in \Pi \). Now given any \( \alpha \in \Delta_+ \), \( \alpha \) is a linear combination of \( \alpha_i \) with positive coefficients. Thus \((\alpha, u) > 0 \). Similarly, \((\alpha, v) > 0 \). Now any point on the straight segment connecting \( u \) and \( v \) has the form \( xu + (1 - x)v \) for some \( 0 \leq x \leq 1 \). Thus it is easy to see that \((\alpha, x) > 0 \). Thus \( u, v \) are in a single component, i.e. an open chamber (we only considered \( \alpha \) being a positive root. But if it is negative, we get similarly that \((\alpha, u), (\alpha, v) \) and \((\alpha, x) \) are all negative).

Now, take any element \( u' \) not in \( C \), then by definition \((\alpha_i, u') \leq 0 \) for some \( \alpha_i \in \Pi \). Then obviously \( u' \) and \( u \in C \) are not in the same component as they are separated by the hyperplane \( T_{\alpha_i} \). Thus \( C \) is an entire component, i.e. an open chamber.

Theorem 21.3. (a) \( W \) permutes all chambers transitively, i.e. for any two chambers \( C_i \) and \( C_j \) there exists \( w \in W \) such that \( w(C_i) = C_j \). (b) Let \( \Delta_+ \) and \( \Delta'_+ \) be two subsets of positive roots, defined by linear functions \( f \) and \( f' \). Then there exists \( w \in W \) such that \( w(\Delta_+) = \Delta'_+ \). In particular, the Cartan matrix of \( (V, \Delta) \) is independent of the choice of \( f \).

Proof. (a) Choose a segment connecting points in \( C_i, C_j \), which doesn’t intersect \( \cup_{\alpha, \beta \in \Delta} (T_\alpha \cap T_\beta) \). Let’s move along the segment until we hit a hyperplane \( T_\alpha \). Then replace \( C_i \) by \( r_\alpha C_i \). After finitely many steps we hit the chamber \( C_j \).

(b) a linear function \( f \) on \( V \) can be written as \( f_\alpha \), where \( f_\alpha(v) = (a, v) \) for fixed \( a \in V \). \( f \) doesn’t vanish on \( \Delta \) means that \( a \notin \cup_{\alpha \in \Delta} T_\alpha \), so \( f = f_\alpha \) with \( a \) in some open chamber. If we move around this chamber, the set \( \Delta_+ \), defined by \( f \) remains unchanged. Hence all the subsets of positive roots in \( \Delta \) are labelled by open chamber and if \( w(C) = C' \), then for the corresponding sets of positive roots \( \Delta_+ \) and \( \Delta'_+ \) we get that \( w(\Delta_+) = \Delta'_+ \).

Definition 21.3. Let \( s_1, ..., s_r \) be the simple reflections in \( W \) (they depend on choice of \( \Delta_+ \)). Any \( w \in W \) can be written as a product \( w = s_{i_1}...s_{i_t} \) due to Theorem 21.2(d). Such a decomposition with minimal possible number of factors \( t \) is called a reduced decomposition and in this case \( t = l(w) \) is called the length of \( w \). Note: \( det w = (-1)^{l(w)} \) since \( det s_i = -1 \). E.g. \( l(1) = 0, l(s_i) = 1, l(s_i, s_j) = 2 \) if \( i \neq j \), but = 0 if \( i = j \) since \( s_i^2 = 1 \).

Lemma 21.4. (Exchange Lemma) Suppose that \( s_{i_1}...s_{i_{k-1}}(\alpha_{i_k}) \in \Delta_-, \alpha_{i_t} \in \Pi \). Then the expression \( w = s_{i_1}...s_{i_t} \) is not reduced. More precisely, \( w = s_{i_1}...s_{i_m} s_{i_{m+1}}...s_{i_t-1} \) for some \( 1 \leq m \leq t - 1 \).

Proof. Consider the roots \( \beta_k = s_{i_k+1}...s_{i_{k-1}}(\alpha_{i_k}) \) for \( 0 \leq k < t - 1 \). Then \( \beta_0 \in \Delta_- \) and \( \beta_{t-1} = \alpha_{i_t} \in \Delta_+ \). Hence there exists \( 1 \leq m \leq t - 1 \) such that \( \beta_m = 0 \), \( \beta_m = \alpha_{i_t} \in \Delta_+ \). But \( \beta_m = s_{i_m} \beta_m \). Hence by Theorem 21.2(a), \( \beta_m = \alpha_{i_m} \in \Pi \). Let \( \overline{w} = s_{i_{m+1}}...s_{i_{t-1}} \), by Ex.21.1(d), it follows \( \overline{w}s_{i_t} = s_{i_t} \overline{w} \). The result follows by multiplying both sides by \( s_{i_1}...s_{i_m} \) on the left.

Corollary 21.5. \( W \) acts simply transitively on chambers, i.e. if \( w(C) = C \), then \( w = 1 \).
Proof. In the contrary case, $w(C) = C$ for some $w \neq 1$, hence $w(\Delta_+) = \Delta_+$ for $\Delta_+$ corresponding to $C$. Take a reduced expression $w = s_{i_1}...s_{i_t}, t \geq 1$. Then $w(\alpha_{i_t}) = s_{i_t}...s_{i_{t-1}}(-\alpha_{i_t}) \in \Delta_+$. Hence $s_{i_1}...s_{i_{t-1}}(\alpha_{i_t}) \in \Delta_-$. Hence by Exchange Lemma, $s_{i_1}...s_{i_t}$ is nonreduced. Contradiction.

Transitivity is proved in Theorem 21.3 (a). \qed