18.745 Introduction to Lie Algebras November 23th, 2010

Lecture 21 — The Weyl Group of a Root System

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Let V be a finite dimensional Euclidean vector space, i.e. a real vector space with a positive definite symmetric bilinear form $(.\,,.)$. Let $a \in V$ be a nonzero vector and denote by r_a the orthogonal reflection relative to a, i.e. $r_a(a) = -a, r_a(v) = v$ if $(a, v) = 0$.

Formula 21.1. $r_a(v) = v - \frac{2(a,v)}{(a,a)}a$.

Exercise 21.1. Prove: (a) $r_a \in \mathcal{O}_v(\mathbb{R})$, i.e. $(r_a u, r_a v) = (u, v), u, v \in V$. (b) $r_a = r_{-a}$ and $r_a^2 = 1$. (c) $det r_a = -1$. (d) If $A \in \mathcal{O}_v(\mathbb{R})$, then $A r_a A^{-1} = r_{A(a)}$.

Proof. For (a), $(r_a u, r_a v) = (u - \frac{2(a,u)}{(a,a)}a, v - \frac{2(a,v)}{(a,a)}a) = (u, v) - \frac{2(a,u)(a,v)}{(a,a)} - \frac{2(a,v)(a,u)}{(a,a)} + 4\frac{(a,u)(a,v)(a,a)}{(a,a)^2} = (u, v)$.

For (b), since $r_a(a) = -a$, we have $r_{-a}(-a) = a$. Also, for any v perpendicular to a, it is also perpendicular to $-a$, thus $r_{-a}(v) = v$. Thus r_a acts the same way as r_{-a} on the entire space, so $r_a = r_{-a}$. Next, $r_a^2(a) = r_a(-a) = a$. Also, for v perpendicular to a, $r_a^2(v) = r_a(v) = v$. Thus, r_a^2 acts the same way as 1 does, so $r_a^2 = 1$.

For (c), r_a fixes the space (of dimension dimV-1) perpendicular to a, thus it has 1 as an eigenvalue of multiplicity $dimV - 1$. The other eigenvalue is -1, corresponding to eigenvector a. Thus $det r_a$ equals product of all eigenvalues, which is -1.

For (d), notice that $A r_a A^{-1}(A(a)) = A r_a(a) = A(-a) = -A(a)$. Also, for any v perpendicular to $A(a)$, $(A^{-1}(v), a) = (v, A(a)) = 0$ as $A \in \mathcal{O}$. Thus $A r_a A^{-1}(v) = A(A^{-1}(v)) = v$. Thus $A r_a A^{-1}$ acts the same way as $r_{A(a)}$, so they are equal. □

Definition 21.1. Let (V, Δ) be a root system. Let W be the subgroup of $\mathcal{O}_v(\mathbb{R})$, generated by all r_α, where $\alpha \in \Delta$. The group W is called the Weyl group of the root system (V, Δ) (and of the corresponding semisimple lie algebra $\mathfrak{g}$).

Proposition 21.1. *(a) $w(\Delta) = \Delta$ for all $w \in W$. (b) W is a finite subgroup of the group $\mathcal{O}_v(\mathbb{R})$.*

Proof. For (a), it suffice to show that $r_\alpha(\beta)(= \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha) \in \Delta$ if $\alpha, \beta \in \Delta$. First, r_α is nonsingular as it has determinant -1. Recall the string property of (V, Δ): $\{\beta - k\alpha | k \in \mathbb{Z}\} \cap (\Delta \cup 0) = \{\beta - p\alpha, ..., \beta + q\alpha\}$, where $p, q \in \mathbb{Z}_+, p - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Hence $p \geq \frac{2(\alpha,\beta)}{(\alpha,\alpha)}, q \geq -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. So if $(\alpha, \beta) \leq 0$, by the string property, we can add α to β at least $-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ times and if $(\alpha, \beta) \geq 0$ we can subtract α from β at least $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ times, which exactly means that $\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Delta \cup \{0\}$. But it can't be 0 since r_α is nonsingular.

(b) is clear since Δ spans V, so if $w \in W$ fixes all elements of Δ, it must be 1, so W embeds in the group of permutations of the finite set Δ by (a). Therefore W is finite. □

Remark 21.1. (a) shows the string property of the root system implies that Δ is W-invariant. One can show converse is true: if we replace string property by W-invariance of Δ and that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$, then we get an equivalent definition of a root system. One has to check only for the case $dimV = 2$. (see Serre)

Fix $f \in V^*$ which doesn't vanish on Δ, let $\Delta_+ = \{\alpha \in \Delta | f(\alpha) > 0\}$ be the subset of positive roots and let $\Pi = \{\alpha_1, ..., \alpha_r\} \in \Delta_+$ be the set of simple roots ($r = dimV$). Then the reflections $s_i = r_{\alpha_i}$ are called simple reflections.

Theorem 21.2. *(a) $\Delta_+ \backslash \{\alpha_i\}$ is s_i-invariant. (b) if $\alpha \in \Delta_+ \backslash \Pi$, then there exists i such that $hts_i(\alpha) < ht\alpha$ ($ht \sum_i k_i \alpha_i = \sum_i k_i$). (c) If $\alpha \in \Delta_+ \backslash \Pi$, then there exists a sequence of simple reflections $s_{i_1}, ... s_{i_k}$ such that $s_{i_1}, ... s_{i_k}(\alpha) \in \Pi$ and also $s_{i_j}, ... s_{i_k}(\alpha) \in \Delta_+$ for all $1 \leq j \leq k$. (d) The group W is generated by simple reflections.*

Proof. (a) For a positive root $\alpha, s_i(\alpha) = \alpha - n\alpha_i$, where n is an integer. If $\alpha \neq \alpha_i$, then all coefficients in the decomposition of simple root remain positive, except possibly for the coefficient of α_i. But in a positive root α, all coefficients are nonnegative. Hence if α is not simple, it must have positive coefficient in front of some simple roots other than α_i (other positive integer multiples of α_i are not in Δ by definition of root system), thus $s_i(\alpha)$ should remain positive.

(b) $s_i(\alpha) = \alpha - \frac{2(\alpha, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i$. Now if $hts_i(\alpha) \geq ht\alpha$ for all i, then $(\alpha, \alpha_i) \leq 0$ for all i. But then $(\alpha, \alpha) = \sum_i a_i(\alpha, \alpha_i) \leq 0$. Since $\alpha = \sum_i a_i \alpha_i, a_i \geq 0$. So $\alpha = 0$, a contradiction.

(c) Just apply (b) finitely many times until we get a simple root.

(d) Denote by W' the subgroup of W, generated by simple reflections. By (c), for any $\alpha \in \Delta_+$ there exists $w \in W'$, such that $w(\alpha) = \alpha_j \in \Pi$. Hence by Ex.21.1(d), $r_\alpha = w^{-1} s_j w$, which lies in W'. So W' contains all reflections r_α with $\alpha \in \Delta$. But $r_{-\alpha} = r_\alpha$, so W' contains all reflections, hence $W' = W$. □

Example 21.1. $\Delta_{A_r} = \{\epsilon_i - \epsilon_j | 1 \leq i, j \leq r+1, i \neq j\} \subset V = \{\sum_{i=1}^{r+1} a_i \epsilon_i | \sum_i a_i = 0, a_i \in \mathbb{R}\} \subset \mathbb{R}^{r+1} = \oplus_{i=1}^{r+1} \mathbb{R}\epsilon_i$, with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$.

We have $r_{\epsilon_i - \epsilon_j}(\epsilon_s) = \epsilon_s - \frac{2(\epsilon_s, \epsilon_i - \epsilon_j)}{2}(\epsilon_i - \epsilon_j) = \begin{cases} \epsilon_s & \text{if} \quad s \neq i, s \neq j \\ \epsilon_j & \text{if} \quad s = i \\ \epsilon_i & \text{if} \quad s = j \end{cases}$ So $r_{\epsilon_i - \epsilon_j}$ is transposition of ϵ_i, ϵ_j, so $W_{A_r} = S_{r+1}$.

Exercise 21.2. Compute Weyl Group for root systems of type B_r, C_r, D_r. In particular show that for B_r and C_r they are isomorphic, but not isomorphic to D_r.

Proof. $\Delta_{B_r} = \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j\} \cup \{\pm\epsilon_i\} \subset \mathbb{R}^r = \oplus_{i=1}^r \mathbb{R}\epsilon_i$, with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. It is easy to see that, as in the example, $r_{\epsilon_i - \epsilon_j}$ switches ϵ_i with ϵ_j, r_{ϵ_i} switches the sign of ϵ_i and the other r_α are generated by the previous two. Thus the Weyl Group consisting of all elements that permute r elements as well as switch some of their signs. So it is the semidirect product group $\mathbb{Z}_2^r \rtimes S_r$. The root system for C_r is just that for B_r with $\pm\epsilon_i$ replaced by $\pm 2\epsilon_i$. So of course the reflections are exactly the same, so Weyl Groups for B_r and C_r are the same.

$\Delta_{D_r} = \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j\} \subset \mathbb{R}^r = \oplus_{i=1}^r \mathbb{R}\epsilon_i$, with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. Now $r_{\epsilon_i - \epsilon_j}$ acts as before, and $r_{\epsilon_i + \epsilon_j}$ switches ϵ_i to $-\epsilon_j$ and ϵ_j to $-\epsilon_i$. Thus Weyl Group is the group of all elements that permute r elements as well as switching an even number of their signs. Thus it is $\mathbb{Z}_2^{r-1} \rtimes S_r$, and it is not isomorphic to that of B_r and C_r. □

Definition 21.2. Consider the open (in usual topology) set $V - \cup_{\alpha\in\Delta} T_\alpha$ in V (T_α is the hyperplane perpendicular to α). The connected components of the set are called open chambers, the closures are called closed chambers. $C = \{v \in V | (\alpha_i, v) > 0, \alpha_i \in \Pi\}$ is called the fundemental chamber; $\overline{C} = \{v | (\alpha_i, v) \geq 0, \alpha_i \in \Pi\}$ is called closed fundemental chamber.

Exercise 21.3. Show that the open fundemental chamber is a chamber.

Proof. We first show that C is connected. Given any $u, v \in C$, we have $(\alpha_i, u) > 0$ for all $\alpha_i \in \Pi$. Now given any $\alpha \in \Delta_+, \alpha$ is a linear comobination of α_i with positive coefficients. Thus $(\alpha, u) > 0$. Similarly, $(\alpha, v) > 0$. Now any point on the straight segment connecting u and v has the form $xu + (1-x)v$ for some $0 \leq x \leq 1$. Thus it is easy to see that $(\alpha, x) > 0$. Thus u, v are in a single component, i.e. an open chamber (we only considered α being a positive root. But if it is negative, we get similarly that $(\alpha, u), (\alpha, v)$ and (α, x) are all negative).

Now, take any element u' not in C, then by definition $(\alpha_i, u') \leq 0$ for some $\alpha_i \in \Pi$. Then obviously u' and $u \in C$ are not in the same component as they are separated by the hyperplane T_{α_i}. Thus C is an entire component, i.e. an open chamber. □

Theorem 21.3. *(a) W permutes all chambers transitively, i.e. for any two chambers C_i and C_j there exists $w \in W$ such that $w(C_i) = C_j$. (b) Let Δ_+ and Δ'_+ be two subsets of positive roots, defined by linear functions f and f'. Then there exists $w \in W$ such that $w(\Delta_+) = \Delta'_+$. In particular, the Cartan matrix of (V, Δ) is independent of the choice of f.*

Proof. (a) Choose a segment connecting points in C_i, C_j, which doesn't intersect $\cup_{\alpha,\beta\in\Delta}(T_\alpha \cap T_\beta)$. Let's move along the segment until we hit a hyperplane T_α. Then replace C_i by $r_\alpha C_i$. After finitely many steps we hit the chamber C_j.

(b) a linear function f on V can be written as f_a, where $f_a(v) = (a, v)$ for fixed $a \in V$. f doesn't vanish on Δ means that $a \notin \cup_{\alpha\in\Delta} T_\alpha$, so $f = f_a$ with a in some open chamber. If we move a around this chamber, the set Δ_+, defined by f remains unchanged. Hence all the subsets of positive roots in Δ are labelled by open chamber and if $w(C) = C'$, then for the corresponding sets of positive roots Δ_+ and Δ'_+ we get that $w(\Delta_+) = \Delta'_+$. □

Definition 21.3. Let $s_1, ...s_r$ be the simple reflections in W (they depend on choice of Δ_+). Any $w \in W$ can be written as a product $w = s_{i_1}...s_{i_t}$ due to Theorem 21.2(d). Such a decomposition with minimal possible number of factors t is called a reduced decomposition and in this case $t = l(w)$ is called the length of w. Note: $det w = (-1)^{l(w)}$ since $det s_i = -1$. E.g. $l(1) = 0, l(s_i) = 1, l(s_i, s_j) = 2$ if $i \neq j$, but $= 0$ if $i = j$ since $s_i^2 = 1$.

Lemma 21.4. *(Exchange Lemma) Suppose that $s_{i_1}...s_{i_{t-1}}(\alpha_{i_t}) \in \Delta_-, \alpha_{i_t} \in \Pi$. Then the expression $w = s_{i_1}...s_{i_t}$ is not reduced. More precisely, $w = s_{i_1}...s_{i_{m-1}}s_{i_{m+1}}...s_{i_{t-1}}$ for some $1 \leq m \leq t-1$.*

Proof. Consider the roots $\beta_k = s_{i_{k+1}}...s_{i_{t-1}}(\alpha_{i_t})$ for $0 \leq k \leq t-1$. Then $\beta_0 \in \Delta_-$ and $\beta_{t-1} = \alpha_{i_t} \in \Delta_+$. Hence there exists $1 \leq m \leq t-1$ such that $\beta_{m-1} \in \Delta_-, \beta_m \in \Delta_+$. But $\beta_{m-1} = s_{i_m}\beta_m$. Hence by Theorem 21.2(a), $\beta_m = \alpha_{i_m} \in \Pi$. Let $\overline{w} = s_{i_{m+1}}...s_{i_{t-1}}$, by Ex.21.1(d), it follows $\overline{w}s_{i_t}\overline{w}^{-1} = s_{i_m}$, or $\overline{w}s_{i_t} = s_{i_m}\overline{w}$. The result follows by multiplying both sides by $s_{i_1}...s_{i_m}$ on the left. □

Corollary 21.5. *W acts simply transitively on chambers, i.e. if $w(C) = C$, then $w = 1$.*

Proof. In the contrary case, $w(C) = C$ for some $w \neq 1$, hence $w(\Delta_+) = \Delta_+$ for Δ_+ corresponding to C. Take a reduced expression $w = s_{i_1}...s_{i_t}, t \geq 1$. Then $w(\alpha_{i_t}) = s_{i_1}...s_{i_{t-1}}(-\alpha_{i_t}) \in \Delta_+$. Hence $s_{i_1}...s_{i_{t-1}}(\alpha_{i_t}) \in \Delta_-$. Hence by Exchange Lemma, $s_{i_1}...s_{i_t}$ is nonreduced. Contradiction.

Transitivity is proved in Theorem 21.3 (a). $\square$