18.745 Introduction to Lie Algebras November 16, 2010

Lecture: 19 Classification of simple finite dimensional Lie algebras over $\mathbb{F}$

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Throughout this lecture $\mathbb{F}$ will be an algebraically closed field of characteristic 0.

Theorem 19.1. *(a) Semisimple Lie algebras over $\mathbb{F}$ are isomorphic if and only if they have the same Dynkin diagram.*
(b) A complete non-redundant list of simple finite dimensional Lie algebras over $\mathbb{F}$ is the following: $\mathfrak{sl}_n(\mathbb{F})$ $(n \geq 2)$, $\mathfrak{so}_n(\mathbb{F})$ $(n \geq 7)$, $\mathfrak{sp}_{2n}(\mathbb{F})$ $(n \geq 2)$ and five exceptions (E_6,E_7,E_8,F_4 and C_2).

Exercise 19.1. Deduce from (a), that the following are isomorphisms: $\mathfrak{sl}_2 \cong \mathfrak{so}_3 \cong \mathfrak{sp}_2$; $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \bigoplus \mathfrak{sl}_2$; $\mathfrak{so}_5 \cong \mathfrak{sp}_4$; $\mathfrak{so}_6 \cong \mathfrak{sl}_4$.
Solution. We know by part (a) of Theorem 19.1 that a semisimple Lie algebras over $\mathbb{F}$ are isomorphic if and only if they have the same Dynkin diagram. Thus we will show that the following are isomorphic to each other by showing their Dynkin Diagrams to be the same. i)$sl_2 \cong so_3 \cong sp_2$.
We know that the Dynkin diagram for $\mathfrak{sl}_2$ is: $\bigcirc$. Similarly, the Dynkin diagram for $\mathfrak{so}_3$ is: $\bigcirc$.
Finally, the Dynkin diagram for $\mathfrak{sp}_2$ is: $\bigcirc$
ii) $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \bigoplus \mathfrak{sl}_2$ Both Dynkian diagrams consist of two disconnected nodes.
iii) $\mathfrak{so}_5 \cong \mathfrak{sp}_4$ Now, the Dynkin diagram for $\mathfrak{so}_5$ is given by: $\bigcirc\!\!\Longrightarrow\!\!\bigcirc$.
The Dynkin diagram for $\mathfrak{sp}_4$ is: $\bigcirc\!\!\Longrightarrow\!\!\bigcirc$.
iv)$\mathfrak{so}_6 \cong \mathfrak{sl}_4$ We know that the Dynkin diagram for $\mathfrak{so}_6$ is: $\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc$.
In the same way, the Dynkin diagram for $\mathfrak{sl}_4$ is: $\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc$.
Thus, $\mathfrak{so}_6 \cong \mathfrak{sl}_4$ by Thm. 19.1.

Proof of Theorem 19.1a) Let $\mathfrak{g}$ be a semisimple finite dimensional Lie algebra over $\mathbb{F}$. Choose a Cartan subalgebra $\mathfrak{h}$ and consider the root space decomposition: $\mathfrak{g} = \mathfrak{h} \bigoplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, where $\Delta \subset \mathfrak{h}^*$ is the set of roots. Choose a linear function f on $\mathfrak{h}^*_\alpha$ which does not vanish on Δ, and let $\Delta_+ = \{\alpha \in \Delta | f(\alpha) > 0\}$ and $\Delta_- = \{\alpha \in \Delta | f(\alpha) < 0\}$.
Then $\Delta = \Delta_+ \coprod \Delta_-$, where $\Delta_- = -\Delta_+$. Let $\Pi \subseteq \Delta_+$ be the set of simple roots. We will prove that nothing depends on f.
Let $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$, which are obviously subalgebras of $\mathfrak{g}$. Then we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \bigoplus \mathfrak{h} \bigoplus \mathfrak{n}_+$$

as vector spaces.

Exercise 19.2. Show that if $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_n$, choosing $\mathfrak{h}$ to be all diagonal matrices in $\mathfrak{g}$; then $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) consists of all strictly upper (resp. lower) triangular matrices in $\mathfrak{g}$.

Solution. For $\mathfrak{sl}_n$, we know that the root space corresponding to $\epsilon_i - \epsilon_j$ is E_{ij} where E_{ij} is 1 in the (i,j) slot and zero elsewhere. In the standard ordering of the roots, $\epsilon_i - \epsilon_j$ is positive if and only if $i < j$. Thus the positive roots spaces correspond to strictly upper triangular matrices and negative root spaces correspond to strictly lower triangular matrices.
For $\mathfrak{so}_n$, we have:

$$\mathfrak{so}_n(\mathbb{F}) = \mathfrak{h} \bigoplus (\bigoplus_{i,j} \mathbb{F} F_{ij}), \tag{2}$$

where $F_{ij} = E_{ij} - E_{n+1-j,n+1-i}$ and $\mathfrak{h} = (a_1, , ...a_r, -a_r, ... - a_1)$. Then the positive roots are: $\epsilon_i - \epsilon_j (i < j)$ and $\epsilon_i + \epsilon_j (i \neq j)$, if $n = 2r$. Thinking of the four r by r blocks in $\mathfrak{so}_n$, the $\epsilon_i + \epsilon_j (i \neq j)$ root spaces fill in the upper right hand block, and the $\epsilon_i - \epsilon_j (i < j)$ fill in the piece above the diagonal for the upper left half. Thus the positive root spaces span the set of upper triangular matrices in $\mathfrak{so}_n$ (which are by definition antisymmetric with respect to the antidiagonal). The odd case is similar.

For $\mathfrak{sp}_n(\mathbb{F})$, we have again the Cartan subalgebra $\mathfrak{h} = (a_1, , ...a_r, -a_r, ... - a_1)$. Here the positive roots are $\epsilon_i + \epsilon_j$ (where i is permitted to equal j) and $\epsilon_i - \epsilon_j$ where $i < j$. We know that $\mathfrak{sp}_n(\mathbb{F})$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $d = -a'$, $b = b'$, $c = c'$ (where $'$ denotes transposition with respect to the antidiagonal). The positive root spaces corresponding to $\epsilon_i - \epsilon_j$ are $F_{ij} = E_{ij} - E_{n+1-j,n+1-i}$ for $1 \leq i, j \leq r$ $(i \leq j)$ - these fill in the part of a above the main diagonal in $\mathfrak{sp}_n(\mathbb{F})$ and also the part of d above this main diagonal by $d = -a'$. The root spaces corresponding to $\epsilon_i + \epsilon_j$ are $F_{ij} = E_{ij} + E_{n+1-j,n+1-i}$ for $1 \leq i \leq r$, $r + 1 \leq j \leq n$, and these spaces fill in the block b. Thus the positive root spaces span the strictly upper triangular matrices.

Continuation of the proof of Theorem 19.1. Recall that for $\alpha \in \Delta$ we can choose $E_\alpha \in \mathfrak{g}_\alpha$ and $F_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $K(E_\alpha, F_\alpha) = 2/K(\alpha, \alpha)$. Then $H_\alpha = 2\nu^{-1}(\alpha)/K(\alpha, \alpha)$ so that $\mathbb{F}E_\alpha + \mathbb{F}F_{-\alpha} + \mathbb{F}H_\alpha$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. Now given $\Pi = (\alpha_1, ..., \alpha_r)$, set $E_i = E_{\alpha_i}$,$F_i = F_{\alpha_i}$,$H_i = H_{\alpha_i}$ for $1 \leq i \leq r$. Then we have:

1) $[H_i, H_j] = 0$ (since the H_i are in a Cartan subalgebra)
2) $[H_i, E_j] = 2(K(\alpha_i, \alpha_j)/K(\alpha_i, \alpha_i))E_j = a_{ij}E_j$, where $a_{ij} = \alpha_j(H_i)$.
3) $[H_i, F_j] = -a_{ij}F_j$
4) $[E_i, F_j] = \delta_{ij}H_j$ (since $\alpha_i - \alpha_j$ is not a root when $i \neq j$ because the α_i are simple)

Definition 19.1. These relations on the E_i, F_i, and H_i are called the Chevalley relations. The E_i, F_i and H_i are called the Chevallay generators.

Lemma 19.1. The E_i (respectively F_i) generate $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$). Consequently the E_i, F_i and H_i generate $\mathfrak{g}$.

Proof. Given $\alpha \in \Delta$, writing $\alpha = \sum_i^r n_i\alpha_i$. We call $\sum_i^r n_i$ the height of the root α. We prove the lemma by induction on the height of a given root. When the height is 1, the root is simple so the E_i certainly generate it. For the inductive step, observe that if $\alpha \in \Delta_+$ but not simple, then for some simple root α_i we have $\mathfrak{g}_\alpha = [\mathfrak{g}_{\alpha-\alpha_i}, \mathfrak{g}_{\alpha_i}]$. Thus we have exhibited $\mathfrak{g}_\alpha$ as a bracket of a term with lower height and one of the E_i, so we are done. □

Definition 19.2. Denote by $\tilde{\mathfrak{g}}(A)$ the Lie algebra with generators E_i, F_i and H_i $(1 \leq i \leq r)$ subject to the Chevalley relations. This Lie algebra is infinite dimensional if $r \geq 1$ but is closely related to $\mathfrak{g}$: we have the surjective homomorphism $\phi : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}$ sending E_i to E_i, F_i to F_i and H_i to H_i.

Lemma 19.2. a) Let $\tilde{\mathfrak{n}}_+$ (resp $\tilde{\mathfrak{n}}_-$) be the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $E_1, ...E_r$ (resp. $F_1, ..., F_r$) and $\mathfrak{h}$ the span of $H_1, ..., H_r$. Then $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{n}}_- \oplus\ \mathfrak{h}$ (direct sum of vector spaces)

b) $\tilde{\mathfrak{n}}_+ = \bigoplus_{\alpha\in\mathbb{Q}_+} \tilde{\mathfrak{g}}_\alpha$ and $\tilde{\mathfrak{n}}_- = \bigoplus_{\alpha\in\mathbb{Q}_+} \tilde{\mathfrak{g}}_{-\alpha}$ where $\mathbb{Q}_+ = \mathbb{Z}_+\Pi/0$

c) If I is an ideal in $\tilde{\mathfrak{g}}(A)$ then $I = (\mathfrak{h} \cap I) \oplus (\bigoplus_{\alpha\in\mathbb{Q}_+\cup-\mathbb{Q}_+} \tilde{\mathfrak{g}}_\alpha \cap I)$.

d) $\tilde{\mathfrak{g}}(A)$ contains a unique proper maximal ideal $I(A)$ provided $D(A)$ is connected.

Proof. For a), we will first show that 1) $[H_i, \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}] \subset \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$, 2) $[E_i, \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}] \subset \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$ and 3) $[F_i, \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}] \subset \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$. Since a general element of $\tilde{\mathfrak{g}}(A)$ is an iterated bracket of the E_i, F_i and H_i, by the Jacobi identity, 1), 2) and 3) together imply that $\tilde{\mathfrak{g}}(A) \subseteq \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$. Therefore $\tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$ is a subalgebra of $\tilde{\mathfrak{g}}(A)$ and must coincide with $\tilde{\mathfrak{g}}(A)$ since it contains all generators. To prove 1), observe that $\tilde{\mathfrak{n}}_+$ is the span of commutators of the form $[E_{i_1}, ..., E_{i_n}]$ so that $[H_i, [E_{i_1}, ...E_{i_n}]] = [[H_i, E_{i_1}], [E_{i_2}, ...E_{i_n}] + [E_{i_1}, [H_i, E_{i_2}], ...E_{i_n}] \in \tilde{\mathfrak{n}}_+$ since $[H_i, E_{i_k}] \in \tilde{\mathfrak{n}}_+$ by the Chevalley relations. The same is true for $\tilde{\mathfrak{n}}_-$. This proves 1). For 2), first observe $[E_{i_1}, ...E_{i_n}] \in \tilde{\mathfrak{n}}_+$. Also we can write $[F_i, [E_{i_1}, ...E_{i_n}]] = [[F_i, E_{i_1}], [E_{i_2}, ...E_{i_n}] + [E_{i_1}, [F_i, E_{i_2}], ...E_{i_n}] + ...$ By the Chevalley relations, $[F_i, E_{i_k}] \in \mathfrak{h}$, so each summand in this decomposition, appealing to the Jacobi identity again, is in $\tilde{\mathfrak{n}}_+$, so $[F_i, [E_{i_1}, ...E_{i_n}]] \in \tilde{\mathfrak{n}}_+$. 3) is established similarly. Thus we have shown $\tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h} = \tilde{\mathfrak{g}}(A)$. To show that the sum is direct, let $h \in \mathfrak{h}$ satisfy $\alpha_i(h) = 1$ for all i (writing $h = \sum x_i H_i$, finding such an h is equivalent to solving the linear system $\sum a_{ij} x_i = 1$, which is possible since $\det A \neq 0$). Then **ad** h acts by positive eigenvalues on $\tilde{\mathfrak{n}}_+$, negative eigenvalues on $\tilde{\mathfrak{n}}_-$, and by 0 on $\mathfrak{h}$. This proves the sum is direct.
To prove d), we'll need a weaker statement than c), namely, that $I = (I\cap\tilde{\mathfrak{n}}_-)\oplus(I\cap\tilde{\mathfrak{n}}_+)\oplus(I\cap\mathfrak{h})$. To prove this we will invoke an earlier lemma which said that for any $\mathfrak{h}$-module $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ and any $\mathfrak{h}$-invariant subspace U, we have $U = \oplus_\lambda U \cap V_\lambda$. Apply this to $\mathfrak{h} = \mathbb{F}h$ and $\pi = \mathbf{ad}$ and $V = \tilde{\mathfrak{g}}(A)$. Since I is an invariant subspace under $\mathfrak{h}$, this gives $I = \bigoplus_\lambda I \cap V_\lambda$ where V_λ is the eigenspace of $\tilde{\mathfrak{g}}(A)$ with height λ. Appealing to the lemma again, the sum of the positive eigenspaces in this decomposition is $\tilde{\mathfrak{n}}_+ \cap I$, the sum of the negative eigenspaces is $\tilde{\mathfrak{n}}_- \cap I$, and zero eigenspace is $\mathfrak{h} \cap I$ establishing $I = (I \cap \tilde{\mathfrak{n}}_-) \oplus (I \cap \tilde{\mathfrak{n}}_+) \oplus (I \cap \mathfrak{h})$.

Let $D(A)$ be connected and I be a proper ideal of $\tilde{\mathfrak{g}}(A)$. Then $I \cap \mathfrak{h} = 0$. Indeed, if $a \in I \cap \mathfrak{h}$ is nonzero, then $\alpha_i(a) \neq 0$ for some i, so that $[a, E_i] = \alpha_i(a)E_i \neq 0$, so $E_i \in I$. Hence $H_i \in I$ by the Chevalley relations. Also by the Chevalley relations, E_j and F_j are contained in I for all j such that $a_{ij} \neq 0$. Since $D(A)$ is connected, it follows that all E_j and F_j are contained in I. By Chevalley relations, this implies the H_i are all contained in I, which would mean $I = \tilde{\mathfrak{g}}(A)$, contradicting the properness of I. Thus we have the decomposition $I = (\tilde{\mathfrak{n}}_+ \cap I) \oplus (\tilde{\mathfrak{n}}_- \cap I)$ for any proper ideal, hence for the sum of all proper ideals, $I(A)$. Hence $I(A)$ is the unique proper, maximal ideal. □

Exercise 19.3. Prove statements b) and c) in the theorem.

Proof. b) Since $\tilde{\mathfrak{n}}_+$ is by definition the span of commutators of the form $[E_{i_1}, ..., E_{i_n}]$, then it is spanned by the subspaces $\tilde{\mathfrak{g}}_\alpha$ where α runs through $\mathbb{Q}_+$ (since each $\tilde{\mathfrak{g}}_\alpha$ consist precisely of terms $[E_{i_1}, ..., E_{i_n}]$ where, if $\alpha = \sum n_i\alpha_i$, each E_i appears n_i times). To see that the sum is direct, first observe that since $\det A \neq 0$, for each j we can find $h_j \in \mathfrak{h}$ such that $\alpha_i(h_j) = \delta_{ij}$. By the Chevalley relations, we have $[h_j, E_i] = \delta_{ij}E_i$. Thus given an iterated bracket in $\tilde{\mathfrak{n}}_+$, $[E_{i_1}, ..., E_{i_n}]$, we have that $[h_j, [E_{i_1}, ..., E_{i_n}]] = c[E_{i_1}, ..., E_{i_n}]$ where c is the number of times j appears in the index set $i_1, ..., i_n$. On each $\tilde{\mathfrak{g}}_\alpha$, $\alpha = \sum n_i\alpha_i$, **ad** h_j acts by the constant n_j. This means each $\tilde{\mathfrak{g}}_\alpha$ is a joint eigenspace for the **ad** h_i, so that sum is direct. The same is true for $\tilde{\mathfrak{n}}_-$.
c) Given the decomposition we already proved, $I = (I\cap\tilde{\mathfrak{n}}_-)\oplus(I\cap\tilde{\mathfrak{n}}_+)\oplus(I\cap\mathfrak{h})$, it is enough to show $I \cap \tilde{\mathfrak{n}}_+ = \bigoplus I \cap \tilde{\mathfrak{g}}_\alpha$. But $\mathfrak{n}_+$ is an $\mathfrak{h}$-module under **ad** and $I \cap \tilde{\mathfrak{n}}_+$ is an invariant subspace since I

is an ideal. Therefore by the lemma we had earlier, $I \cap \tilde{\mathfrak{n}}_+ = \bigoplus I \cap \tilde{\mathfrak{g}}_\alpha$ since by b), $\mathfrak{n}_+ = \bigoplus \tilde{\mathfrak{g}}_\alpha$ is a decomposition of $\tilde{\mathfrak{n}}_+$ by weights of $\mathbf{ad}\ \mathfrak{h}$ acting on $\tilde{\mathfrak{n}}_+$. □

Continuation of the proof of Theorem 19.1. By part d) of the lemma $I(A)$ is the unique proper maximal ideal in $\tilde{\mathfrak{g}}(A)$. Now set $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/I(A)$. Since $\mathfrak{g}$ is simple, ker $(\phi : \tilde{\mathfrak{g}}(A) \to \mathfrak{g})$ is a maximal proper ideal. By Lemma 2d, ker $\phi = I(A)$. Hence ϕ induces an isomorphism between $\mathfrak{g}(A)$ and $\mathfrak{g}$. This proves the "if" part of the Theorem 19.1a. The "only if" part will follow once we show the independence of A from the choice of f.

So far, we have shown that if $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_m, \mathfrak{sp}_n$ then $\mathfrak{g} \cong \mathfrak{g}(A)$ where A is the Cartan matrix of $\mathfrak{g}$. The only remaining simple, finite dimensional Lie algebras can be $\mathfrak{g}(A)$ where $A = E_6, E_7, E_8, F_4, G_2$. Hence to complete part b) of the theorem, we need to prove that the dimension of $\mathfrak{g}(A)$ is finite in these five cases. We will prove this by exhibiting explicit constructions of these five.