18.745 Introduction to Lie Algebras	November 04, 2010

Lecture 17 — Cartan Matrices and Dynkin Diagrams

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Previously, given a semisimple Lie algebra $\mathfrak{g}$ we constructed its associated root system (V, Δ). (The construction depends on choosing a Cartan subalgebra, but by Chevalley's theorem, the root systems constructed from the same $\mathfrak{g}$ are isomorphic.) Next, given a root system we'll construct a *Cartan matrix* A, and from this we'll eventually see how to reconstruct $\mathfrak{g}$.

We'll see that to every root system there corresponds a semisimple Lie algebra, so it's important to know all the root systems. Last time we saw the four series A_r, B_r, C_r, and D_r, and the three exceptions E_6, E_7, and E_8. The remaining two exceptions are F_4 and G_2, which we will describe in the following exercises.

Exercise 17.1. Define:

$$V = \bigoplus_{i=1}^{4} \mathbb{R}\epsilon_i; \qquad (\epsilon_i, \epsilon_j) = \delta_{ij};$$
$$Q_{F_4} = \left\{ \sum_{i=1}^{4} a_i\epsilon_i \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\}; \text{ and}$$
$$\Delta_{F_4} = \{\alpha \in Q_{F_4} \mid (\alpha, \alpha) = 1 \text{ or } 2\}.$$

Show that (V, Δ_{F_4}) is an indecomposable root system of rank 4 with 48 roots.

Solution. A simple verification shows $\Delta = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : i \neq j\} \cup \{1/2(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$, which has 48 elements.

All that is difficult is to check that the string property holds. For this, we tabulate enough α, β with the corresponding data $p, q, (\alpha, \beta), (\alpha, \alpha)$. I will use a shorthand for writing vectors — a string of four numbers $\eta_1\eta_2\eta_3\eta_4$ represents the vector $\eta_1\varepsilon_1 + \eta_2\varepsilon_2 + \eta_3\varepsilon_3 + \eta_4\varepsilon_4$. Depending on how many numbers are presumed zero in such a string, we take the other numbers η to be ± 1 or $\pm 1/2$ as appropriate.

α	β	p	q	$p - q$	(α, β)	(α, α)
1000	$\eta_1\eta_2 00$	$1 + \eta_1$	$1 - \eta_1$	$2\eta_1$	η_1	1
	$0\eta_2\eta_3 0$	0	0	0	0	1
	$\eta_1\eta_2\eta_3\eta_4$	$1/2 + \eta_1$	$1/2 - \eta_1$	$2\eta_1$	η_1	1
1100	$\eta_1 000$	$1/2(1 + \eta_1)$	$1/2(1 - \varepsilon_1)$	η_1	η_1	2
	$00\eta_3 0$	0	0	0	0	2
	$\eta_1 0\eta_3 0$	$1/2(1 + \eta_1)$	$1/2(1 - \eta_1)$	η_1	η_1	2
	$\eta_1\eta_2\eta_3\eta_4$	$\delta_{\eta_1=\eta_2=1/2}$	$\delta_{\eta_1=\eta_2=-1/2}$	$\eta_1 + \eta_2$	$\eta_1 + \eta_2$	2
$1/2(1111)$	$\eta_1 000$	$1/2(1 + \eta_1)$	$1/2(1 - \eta_1)$	η_1	$1/2\,\eta_1$	1
	$\eta_1\eta_2 00$	$\delta_{\eta_1=\eta_2=1}$	$\delta_{\eta_1=\eta_2=-1}$	$1/2(\eta_1 + \eta_2)$	$1/2(\eta_1 + \eta_2)$	1
	$\eta_1\eta_2\eta_3\eta_4$	$2\delta_{4+} + \delta_{3+} + \delta_{2+}$	$2\delta_{0+} + \delta_{1+} + \delta_{2+}$	$\sum \eta_i$	$\sum \eta_i$	1

In the last line here we have written δ_{k+} to mean 1 when there are exactly k positive η_i and zero otherwise. While considering each calculation of p and q, that the roots actually appear in strings without gaps is easily checked. Thus we have a root system. To see that it is indecomposable, we need to note that all the roots are equivalent under the the equivalence relation generated by $\alpha \sim \alpha'$ when $(\alpha, \alpha') \neq 0$. Note that $\pm\varepsilon_i \sim 1/2(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$, so that $\{\varepsilon_i, 1/2(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$ is contained in an equivalence class. Next, $\pm\varepsilon_i \pm \varepsilon_j \sim \varepsilon_i$, showing that all roots are equivalent. Thus the root system is indecomposable.

Exercise 17.2. Show that the following describes an indecomposable root system with 12 roots:

$$\begin{aligned} &V_{G_2} = V_{A_2}; \\ &Q_{G_2} = Q_{A_2}; \text{ and} \\ &\Delta_{G_2} = \{\alpha \in Q_{A_2} \mid (\alpha, \alpha) = 2 \text{ or } 6\}. \end{aligned}$$

Solution. $\Delta = \{(a_1, a_2, a_3) \mid a_1 + a_2 + a_3 = 0,\ a_i \in \mathbb{Z},\ \sum a_i^2 \in \{2, 6\}\}$. Now in order to have square sum 2, exactly two of the a_i must equal ± 1, so we have $\pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1)$. In order to have square sum 6, exactly one must be ± 2, and the other two must both be ∓ 1: $\pm(2, -1, -1), \pm(-1, 2, -1), \pm(-1, -1, 2)$.

We tabulate enough of the relevant quantities below to verify that G_2 is a root system, in the style of the previous exercise. (Here ε stands for 1 or -1.)

α	β	p	q	$p-q$	(α,β)	(α,α)
$(1,-1,0)$	$(\varepsilon,-\varepsilon,0)$	$1+\varepsilon$	$1-\varepsilon$	2ε	2ε	2
	$(\varepsilon,0,-\varepsilon)$	$1/2(3+\varepsilon)$	$1/2(3-\varepsilon)$	ε	ε	2
	$(2\varepsilon,-\varepsilon,-\varepsilon)$	$3/2(1+\varepsilon)$	$3/2(1+\varepsilon)$	3ε	3ε	2
	$(-\varepsilon,-\varepsilon,2\varepsilon)$	0	0	0	0	2
$(2,-1,-1)$	$(\varepsilon,-\varepsilon,0)$	$1/2(1+\varepsilon)$	$1/2(1-\varepsilon)$	ε	3ε	6
	$(0,\varepsilon,-\varepsilon)$	0	0	0	0	6
	$(2\varepsilon,-\varepsilon,-\varepsilon)$	$1+\varepsilon$	$1-\varepsilon$	2ε	6ε	6
	$(-\varepsilon,2\varepsilon,-\varepsilon)$	$1/2(1-\varepsilon)$	$1/2(1+\varepsilon)$	$-\varepsilon$	-3ε	6

Finally, to see that G_2 is irreducible, note that no two of the shorter roots are perpendicular, and each of the longer roots is not perpendicular to one of the shorter roots.

Definition 17.1. Suppose (V, Δ) is a root system and $f : V \to \mathbb{R}$ is a linear map such that $f(\alpha) \neq 0$ for all $\alpha \in \Delta$. Then:

(i) $\alpha \in \Delta$ is *positive* if $f(\alpha) > 0$ and *negative* if $f(\alpha) < 0$.

(ii) A positive root is *simple* if it cannot be written as the sum of two positive roots.

(iii) A *highest root* $\theta \in \Delta$ is a root where f is maximal; that is, $f(\theta) \geq f(\alpha)$ for all $\alpha \in \Delta$.

Notation 17.1.

- Δ_+ is the set of positive roots, and Δ_- is the set of negative roots.
- $\Pi \subset \Delta_+$ is the set of simple roots.
- Π is *indecomposable* if it can't be written as a disjoint union of orthogonal sets $\Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$.

Theorem 17.1 (Dynkin). *(a) If $\alpha, \beta \in \Pi$ and $\alpha \neq \beta$, then $\alpha - \beta \notin \Delta$ and $(\alpha, \beta) \leq 0$.*

(b) *Every positive root is a nonnegative integer linear combination of simple roots; i.e.* $\Delta_+ \subseteq \mathbb{Z}_{\geq 0}\Pi$.

(c) *If* $\alpha \in \Delta_+ \setminus \Pi$ *then* $\alpha - \gamma \in \Delta$ *for some* $\gamma \in \Delta$*; moreover, then* $\alpha - \gamma \in \Delta_+$.

(d) Π *is a basis of* V *over* $\mathbb{R}$ *and of the lattice* Q *over* $\mathbb{Z}$*. Hence the integer linear combinations from part (b) are unique.*

(e) Δ *is indecomposable if and only if* Π *is indecomposable.*

Proof. (a) This is a proof by contradiction. If $\alpha - \beta = \gamma \in \Delta$, then either

- $\gamma \in \Delta_+$, so $\alpha = \beta + \gamma$, which contradicts $\alpha \in \Pi$; or
- $\gamma \in \Delta_-$, so $\beta = \alpha + (-\gamma)$, which contradicts $\beta \in \Pi$.

(b) If α is simple then we're done. Otherwise, $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Delta_+$. Then $f(\alpha) = f(\beta) + f(\gamma)$, so both $f(\beta)$ and $f(\gamma)$ are strictly less than $f(\alpha)$. Repeat this process with β and γ until all summands are simple (which must happen in finitely many steps since Δ is finite), thus yielding α as a sum of simple roots.

(c) Suppose $\alpha \in \Delta_+ \setminus \Pi$. If $\alpha - \gamma \notin \Delta$ for all $\gamma \in \Pi$, then the string condition would imply $\frac{2(\gamma,\alpha)}{(\gamma,\gamma)} \leq 0$ for all $\gamma \in \Pi$. Then by (b),

$$(\alpha, \alpha) = \left(\alpha, \sum_{\gamma \in \Pi} k_\gamma \gamma\right) \leq 0,$$

which would imply $\alpha = 0$ and thus $\alpha \notin \Delta$. Hence $\alpha - \gamma \in \Delta$ for some $\gamma \in \Pi \subset \Delta$.

Now if $\alpha - \gamma = \beta \in \Delta_-$, then $\gamma = \alpha + (-\beta)$, which would contradict γ being simple. Therefore $\alpha - \gamma \in \Delta_+$.

(d) From (b) we have that Π spans Δ_+ over $\mathbb{Z}$. Then since $\Delta = \Delta_+ \cup \Delta_- = \Delta_+ \cup -(\Delta_+)$ and $Q = \mathbb{Z}\Delta$, Π spans Q over $\mathbb{Z}$ and thus spans V over $\mathbb{R}$.

To prove linear independence of Π, suppose the contrary—that there existed a nontrivial linear combination of simple roots $\sum_i k_i \alpha_i = 0$. Split this into positive and negative parts, moving the negative parts to the other side to obtain

$$\gamma := \sum_i a_i \alpha_i = \sum_i b_i \alpha_i,$$

where all a_i and b_i are nonnegative and $a_i b_i = 0$ for all i. Since all α_i are positive, $f(\gamma) > 0$, so $\gamma \neq 0$ and $(\gamma, \gamma) > 0$. However, by (a) we also have

$$(\gamma, \gamma) = \left(\sum a_i \alpha_i, \sum b_j \alpha_j\right) \leq 0,$$

thus giving us a contradiction.

(e) If (V, Δ) is decomposable, then by (d), so is Π. Conversely, if Π decomposes as $\Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$, then we will show $\Delta = (\mathbb{Z}\Pi_1 \cap \Delta) \cup (\mathbb{Z}\Pi_2 \cap \Delta)$.

Suppose the contrary—then $\alpha = \gamma_1 + \gamma_2$ for some $\alpha \in \Delta$ and $\gamma_i \in \mathbb{Z}_{\geq 0}\Pi_i$. By flipping the sign of α if necessary, we can assume $\alpha \in \Delta_+$. By (b), we can subtract simple roots until γ_1 is simple. Then $\Pi_1 \perp \Pi_2$ implies

$$\frac{2(\alpha, \gamma_1)}{(\gamma_1, \gamma_1)} = \frac{2(\gamma_1, \gamma_1)}{(\gamma_1, \gamma_1)} = 2,$$

so by the string property $\beta := \alpha - 2\gamma_1 = \gamma_2 - \gamma_1$ is a root (it can't be zero since $\Pi_1 \perp \Pi_2$).

Flipping the sign of β if necessary, we can assume $\beta \in \Delta_+$. However, the decomposition $\beta = \gamma_2 - \gamma_1$ (or $\gamma_1 - \gamma_2$ if we flipped the sign) can be made into an integer linear combination of simple roots with mixed signs (by expanding γ_2 in terms of simple roots). By (d), this linear combination is unique, so the nonnegative linear combination guaranteed by (b) cannot exist. □

Exercise 17.3. Prove that if (V, Δ) is an indecomposable root system and $f : V \to \mathbb{R}$ is a linear map such that $f(\alpha) \neq 0$ for all $\alpha \in \Delta$, then there exists a unique highest root $\theta \in \Delta$.

Solution. Suppose first that $\theta = \sum \lambda_i \alpha_i$ is a highest root. We wish first to show that $\lambda_i > 0$ for all i (it is a basic property of root systems that $\lambda_i \geq 0$ for all i). Suppose on the contrary that $\lambda_j = 0$. Expanding $(\alpha_j, \theta) = \sum \lambda_i(\alpha_j, \alpha_i)$, as all the simple roots are at obtuse or right angles, we have $(\alpha_j, \theta) \leq 0$, with equality iff $\alpha_j \perp \alpha_i$ whenever $\lambda_i \neq 0$. Now as $\theta + \alpha_j$ cannot be a root, the string condition implies that $(\alpha_j, \theta) = 0$. Thus $\alpha_j \perp \alpha_i$ whenever $\lambda_i \neq 0$ and $\lambda_j = 0$. Thus, if $\lambda_j = 0$ for some j, the simple roots decompose into disjoint perpendicular subsets $\{\alpha_j : \lambda_j = 0\}$ and $\{\alpha_j : \lambda_j \neq 0\}$. This is impossible, as the root system is indecomposable.

Now suppose we have two distinct highest roots θ and $\tilde{\theta}$. Then $(\theta, \tilde{\theta}) \leq 0$, as otherwise $\theta - \tilde{\theta}$ is a root (yet $f(\theta - \tilde{\theta}) = 0$ which is impossible). Thus $0 \geq (\theta, \tilde{\theta}) = \sum \lambda_i(\alpha_i, \tilde{\theta})$, showing that $(\alpha_i, \tilde{\theta}) = 0$ (as the string condition implies that $(\alpha_i, \tilde{\theta}) \geq 0$ for each i). Yet then $\tilde{\theta}$ is perpendicular to a basis of V, so $\theta = 0$, a contradiction.

Definition 17.2. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots of Δ (corresponding to f). The matrix $A = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right)_{ij=1}^r$ is called the *Cartan matrix*. We will show later that it is independent of choice of f.

Proposition 17.2. *The Cartan matrix has all entries integers, and the following properties:*

(a) $A_{ii} = 2$ *for all* i.

(b) If $i \neq j$ *then* $A_{ij} \leq 0$, *and* $A_{ij} = 0 \iff A_{ji} = 0$.

(c) All principle values of A *are positive. In particular,* $\det A > 0$.

Proof. (a) is immediate, and (b) follows by Theorem 17.1(a). For (c), we note that we may factorise A as $\operatorname{diag}\left(\frac{2}{(\alpha_i, \alpha_i)}\right)_{i=1}^r \cdot ((\alpha_i, \alpha_j))_{ij=1}^r$. The first term may be ignored, while the second is the Gram matrix for the inner product with respect to Π. The result follows by Sylvester's criterion. □

Definition 17.3. If $V, \Delta)$ is indecomposable, we have a unique highest root θ (by the above exercise). Let $\alpha_0 = -\theta$, and $\Pi_0 = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$. The matrix $\widetilde{A} = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right)_{ij=0}^r$ is called the *extended Cartan matrix*.

Exercise 17.4. $\widetilde{A}$ satisfies all of the properties of Proposition 17.2, except $\det \widetilde{A} = 0$.

Solution. The proof of properties (a) and (b) are exactly the same as for the standard Cartan matrix A. However, we need to see tht the principle minors are still all positive, except for the determinant (which is zero).

Again, $\tilde{A}$ factors as $\tilde{A} = \operatorname{diag}(2/(\alpha_i, \alpha_i))_{i=0}^n \cdot ((\alpha_i, \alpha_j))_{ij=0}^n$, and the first matrix has all diagonal entries positive, so we only need to investigate the principle minors of $Q = ((\alpha_i, \alpha_i))_{i=0}^n$.

Given any proper subset I of $\{0, \ldots, n\}$, I is a subset of some subset I' of $\{0, \ldots, n\}$ with n elements. Now the α_i with $i \in I'$ are a basis of V, as the highest root is $\sum \lambda_i \alpha_i$ with the λ_i nonzero. Thus the submatrix of Q corresponding to I' is a Gram matrix for the inner product, and thus has all principle minors zero. In particular, the principle minor of Q corresponding to I is nonzero (by Sylvester's criterion), and thus so is the corresponding principle minor of $\tilde{A}$. Of course, the determinant of Q (and thus A) is zero as the α_i ($i = 0, \ldots, n$) are not linearly independent.

Definition 17.4. A $r \times r$ matrix satisfying all of the properties of Proposition 17.2 is called an *abstract Cartan matrix.*

Let's classify the abstract Cartan matrices. The only 1×1 such matrix is (2), the Cartan matrix of type A_1. There are more possibilities for 2×2 abstract Cartan matrices A. We know that $A = \left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ for nonnegative integers a, b. Moreover, $4 - ab > 0$, so that $ab = 3$. There are four possibilities for A (up to taking the transpose):

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

The Dynkin diagram $D(A)$ depicts the Cartan matrix A by a graph with r vertices in bijection with the simple roots. For any two distinct simple roots α_i and α_j, the corresponding 2×2 Cartan matrix will be one of the above four, or a transpose thereof. The vertices corresponding to a chosen pair of roots are joined as follows in each case:

$$\begin{array}{cccccc} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \\ \circ \quad \circ & \circ\!-\!\!\!-\!\circ & \circ\!\Rightarrow\!\circ & \circ\!\Leftarrow\!\circ & \circ\!\Rrightarrow\!\circ & \circ\!\Lleftarrow\!\circ \end{array}$$

In each diagram, the left node corresponds to the first row and column of the matrix. Note that when the two roots in question have different lengths, the arrow points to the shorter root. The diagram formed in this way is the *Dynkin diagram* $D(A)$.

Remark. Any subdiagram of a Dynkin diagram is again a Dynkin diagram. Π is indecomposable if and only if the Dynkin diagram is connected.

We can now calculate Cartan matrices and Dynkin diagrams for various root systems. In each of the following root systems, we have $\{\varepsilon_i\}$ an orthonormal set of vectors (which is enough to calculate inner products).

For A_r, $\Delta = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq r+1, i \neq j\}$. Letting $f(\varepsilon_i) = r + 1 - i$, the simple roots may be indentified easily, as they all take value 1 under f. We find:

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq r+1\} \supset \Pi = \{e_i - e_{i+1} : 1 \leq i \leq r\}, \text{ and } \theta = \varepsilon_1 - \varepsilon_{r+1}.$$

For the rest of the root systems treated in this lecture, we use the same function f (although there is no basis vector ε_{r+1}).

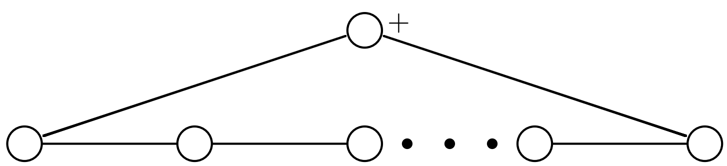
For B_r, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i : 1 \le i, j \le r, i \ne j\}$. We find:

$$\Pi = \{e_i - e_{i+1} : 1 \le i \le r-1\} \cup \{\varepsilon_r\}, \text{ and } \theta = \varepsilon_1 + \varepsilon_2.$$

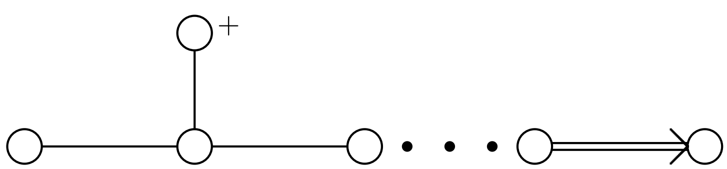
The Cartan matrices are shown below when $r = 6$ — the pattern can be read off easily enough. The augmented Cartan matrices are the whole matrix, where the standard matrix is the bottom right 6×6 block.

$$A_6 : \left(\begin{array}{c|cccccc} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ \hline -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right) \qquad B_6 : \left(\begin{array}{c|cccccc} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{array}\right)$$

A_r :



B_r :



These diagrams are the extended Dynkin diagrams, while the Dynkin diagrams are obtained by removing the node marked with +. (This comment applies to all four Dynking diagrams shown.)

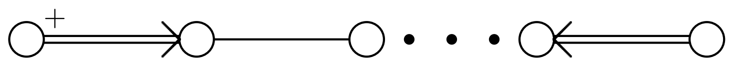
Exercise 17.5. Perform the same analysis for the root systems C_r ($r \ge 2$) and D_r ($r \ge 3$).

Solution. For C_r, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid i \ne j\}$, $\Delta_+ = \{\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j \mid i \ne j\} \cup \{2\varepsilon_i\}$ and $\Pi = \{\alpha_i\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($1 \le i < r$) and $\alpha_r = 2\varepsilon_r$. Furthermore, $\alpha_0 = -2\varepsilon_1$.

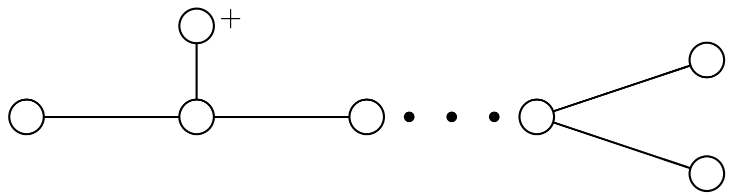
For D_r, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \ne j\}$, $\Delta_+ = \{\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j \mid i \ne j\}$ and $\Pi = \{\alpha_i\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($1 \le i < r$) and $\alpha_r = \varepsilon_{r-1} + \varepsilon_r$. Furthermore, $\alpha_0 = -\varepsilon_1 - \varepsilon_2$.

$$C_6 : \left(\begin{array}{c|cccccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right) \qquad D_6 : \left(\begin{array}{c|cccccc} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{array}\right)$$

C_r :



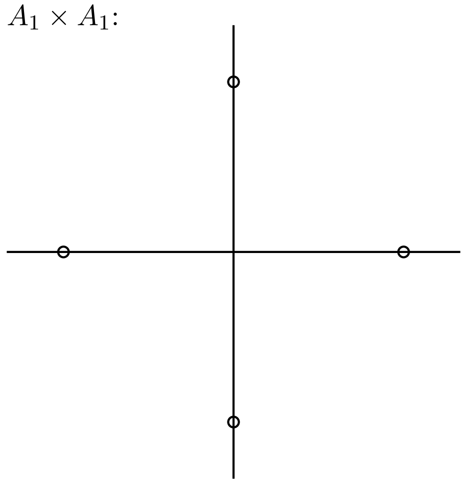
D_r :



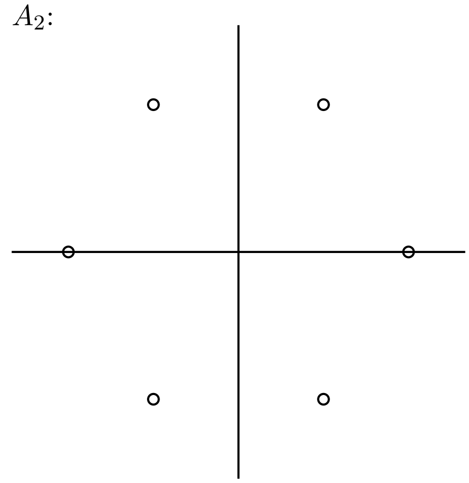
Exercise 17.6. Draw on the plane the root systems $A_1 \times A_1$, A_2, B_2 and G_2.

Solution.

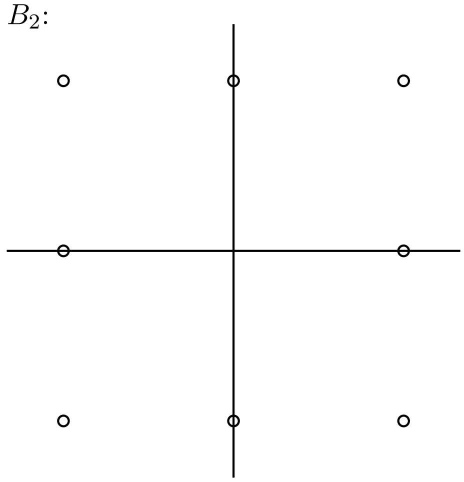
$A_1 \times A_1$:



A_2:



B_2:



G_2: