18.745 Introduction to Lie Algebras October 21, 2010

Lecture 13 — Structure Theory of Semisimple Lie Algebras II

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Throughout this lecture, let $\mathfrak{g}$ be a finite dimensional semisimple Lie Algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0.

So far, we have proved:

1. The Killing form K of $\mathfrak{g}$ is non-degenerate.
2. The algebra $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{h}$. Furthermore, $\mathfrak{h}$ is abelian and diagonalizable on $\mathfrak{g}$, and we have:
$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right),$$
where,
$$\begin{aligned} \mathfrak{g}_\alpha &= \{a \in \mathfrak{g} | [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\}, \\ \Delta &= \{\alpha \in \mathfrak{h}^* | \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}, \\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] &\subseteq \mathfrak{g}_{\alpha+\beta}, \text{ which is 0 if } \alpha + \beta \notin \Delta \cup \{0\}, \text{ where } \mathfrak{g}_0 = \mathfrak{h}. \end{aligned}$$
3. The restriction $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$, *i.e.* $K(a, b)$ with $a \in \mathfrak{g}_\alpha$ and $b \in \mathfrak{g}_{-\alpha}$, is non-degenerate, so it induces a pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. In particular, we have $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$.
4. The restriction $K|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, hence we have an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ given by $\nu(h)(h') = K(h, h')$ for all $h, h' \in \mathfrak{h}$. The map ν induces a bilinear form on $\mathfrak{h}^*$ by $K(\alpha, \beta) = \beta\left(\nu^{-1}(\alpha)\right) = \alpha\left(\nu^{-1}(\beta)\right)$ for all $\alpha, \beta \in \mathfrak{h}^*$. We proved that $K(\alpha, \alpha) \neq 0$ if $\alpha \in \Delta$.
5. For all $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$, we have:
$$[e, f] = K(e, f)\nu^{-1}(\alpha).$$

Now, given $\alpha \in \Delta$, pick non-zero $E \in \mathfrak{g}_\alpha$ and $F \in \mathfrak{g}_{-\alpha}$ such that $K(E, F) = \frac{2}{K(\alpha,\alpha)}$. Let $H = \frac{2\nu^{-1}(\alpha)}{K(\alpha,\alpha)}$. Then, we can check that:

$$\begin{aligned} [H, E] &= 2E, \\ [H, F] &= -2F, \\ [E, F] &= H. \end{aligned}$$

The choice of E and F is possible by 3 and the last claim of 4. We only verify the first equality, the second being analogous and the third coming from 5. We have:

$$[\frac{2\nu^{-1}(\alpha)}{K(\alpha, \alpha)}, E] = \frac{2\alpha\left(\nu^{-1}(\alpha)\right)}{K(\alpha, \alpha)}E = \frac{2K(\alpha, \alpha)}{K(\alpha, \alpha)}E = 2E,$$

where the first equality comes from $\nu^{-1}(\alpha) \in \mathfrak{h}$ and $E \in \mathfrak{g}_\alpha$.

If we now let $\mathfrak{a}_\alpha = \mathbb{F}E + \mathbb{F}F + \mathbb{F}H$, then $\mathfrak{a}_\alpha$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ via:

$$E \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 1.1 (Key Lemma for $\mathfrak{sl}_2(\mathbb{F})$)**.** *Let $\mathbb{F}$ be a field of characteristic 0. Let π be a representation of $\mathfrak{sl}_2(\mathbb{F})$ in a vector space V over $\mathbb{F}$ and let $v \in V$ be a non-zero vector such that $\pi(E)v = 0$ and $\pi(H)v = \lambda v$ for some $\lambda \in \mathbb{F}$ (such vector is called a singular vector of weight λ).*

Then:

a) $\pi(H)\pi(F)^n v = (\lambda - 2n)\pi(F)^n v$ *for any* $n \in \mathbb{Z}_{\geq 0}$.

b) $\pi(E)\pi(F)^n = n(\lambda - n + 1)\pi(F)^{n-1} v$ *for any* $n \in \mathbb{Z}_{\geq 1}$.

c) *If* $\dim V < \infty$*, then* $\lambda \in \mathbb{Z}_{\geq 0}$*, the vectors* $\pi(F)^j v$ *for* $0 \leq j \leq \lambda$ *are linearly independent, and* $\pi(F)^{\lambda+1} v = 0$.

Proof. **a)** We prove this by induction on n. For $n = 0$, the result is given to us. Suppose it holds for $n = k - 1$ for some $k > 0$. Then:

$$\begin{aligned} \pi(H)\pi(F)^k v &= \pi(F)\pi(H)\pi(F)^{k-1} v + [\pi(H), \pi(F)]\pi(F)^{k-1} v \\ &= \pi(F)\left(\lambda - 2(k-1)\right)\pi(F)^{k-1} v + \pi\left([H, F]\right)\pi(F)^{k-1} v \\ &= \left(\lambda - 2(k-1)\right)\pi(F)^k v + \pi(-2F)\pi(F)^{k-1} v \\ &= \left(\lambda - 2(k-1)\right)\pi(F)^k v - 2\pi(F)^k v \\ &= \left(\lambda - 2k\right)\pi(F)^k v. \end{aligned}$$

This completes the induction.

Exercise 13.1. b) Again, we use induction on n. For the case $n = 1$, we have $\pi(E)\pi(F)v = \pi(F)\pi(E)v + [\pi(E), \pi(F)]v = 0 + \pi\left([E, F]\right)v = \pi(H)v = \lambda v$. Suppose the result holds for $n = k - 1$ for some $k > 1$. Then:

$$\begin{aligned} \pi(E)\pi(F)^k v &= \pi(F)\pi(E)\pi(F)^{k-1} v + [\pi(E), \pi(F)]\pi(F)^{k-1} v \\ &= \pi(F)(k-1)\left(\lambda - (k-1) + 1\right)\pi(F)^{k-2} v + \pi\left([E, F]\right)\pi(F)^{k-1} v \\ &= (k-1)\left(\lambda - (k-1) + 1\right)\pi(F)^{k-1} v + \pi(H)\pi(F)^{k-1} v \\ &= (k-1)\left(\lambda - (k-1) + 1\right)\pi(F)^{k-1} v + \left(\lambda - 2(k-1)\right)\pi(F)^{k-1} v \\ &= k(\lambda - k + 1)\pi(F)^{k-1} v. \end{aligned}$$

This completes the proof.

c) Suppose $\lambda \notin \mathbb{Z}_{\geq 0}$. Then, the term $n(\lambda - n + 1)$ is non-zero for all $n \geq 1$. Hence, by induction, we get $\pi(F)^n v \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$. But by **a)**, this implies that all vectors $\pi(F)^n v$ are eigenvectors of $\pi(H)$ with distinct eigenvalues. This allows us to conclude $\dim V = \infty$, therefore proving the first claim. If now, $\lambda \in \mathbb{Z}_{\geq 0}$, by the same argument we see that the vectors $\pi(F)^n v$ are linearly independent for $0 \leq n \leq \lambda$, and moreover, if $\pi(F)^{\lambda+1} v \neq 0$, then by induction we see that $\pi(F)^n v \neq 0$ for all $n > \lambda + 1$, and so there are infinitely many linearly independent vectors. Hence, if $\dim V < \infty$, then $\pi(F)^{\lambda+1} v = 0$, proving the last two claims.

□

Exercise 13.2. Using the notation of the Key Lemma for $\mathfrak{sl}_2(\mathbb{F})$, if v instead satisfies that $\pi(F)v = 0$ and $\pi(H)v = \lambda v$, then we have:

a) $\pi(H)\pi(E)^n v = (\lambda + 2n)\pi(E)^n v$ for any $n \in \mathbb{Z}_{\geq 0}$.

b) $\pi(F)\pi(E)^n = -n(\lambda + n - 1)\pi(E)^{n-1} v$ for any $n \in \mathbb{Z}_{\geq 1}$.

c) If $\dim V < \infty$, then $-\lambda \in \mathbb{Z}_{\geq 0}$, the vectors $\pi(E)^j v$ for $0 \leq j \leq -\lambda$ are linearly independent, and $\pi(F)^{-\lambda+1} v = 0$.

Proof. It is enough to check that the function $\psi(E) = F$, $\psi(F) = E$, $\psi(H) = -H$, is an automorphism of $\mathfrak{a}_\alpha$. Indeed:

$$[\psi(H), \psi(E)] = 2\psi(E),$$
$$[\psi(H), \psi(F)] = -2\psi(F),$$
$$[\psi(E), \psi(F)] = \psi(H).$$

Thus, the Key Lemma shows that if $\pi(\psi(E))v = 0$ and $\pi(\psi(H))v = \lambda' v$ with $\lambda' \in \mathbb{F}$, then:

a) $\pi(\psi(H))\pi(\psi(F))^n v = (\lambda' - 2n)\pi(\psi(F))^n v$ for any $n \in \mathbb{Z}_{\geq 0}$.

b) $\pi(\psi(E))\pi(\psi(F))^n = n(\lambda' - n + 1)\pi(\psi(F))^{n-1} v$ for any $n \in \mathbb{Z}_{\geq 1}$.

c) If $\dim V < \infty$, then $\lambda' \in \mathbb{Z}_{\geq 0}$, the vectors $\pi(\psi(F))^j v$ for $0 \leq j \leq \lambda'$ are linearly independent, and $\pi(\psi(F))^{\lambda'+1} v = 0$.

Now, let $\lambda' = -\lambda$ and evaluate ψ to obtain the result. □

Theorem 1.2. *The root space decomposition of $\mathfrak{g}$ with respect to a Cartan Subalgebra $\mathfrak{h}$ and the set of roots Δ satisfy the following properties:*

a) $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Delta$.

b) *If* $\alpha, \beta \in \Delta$, *then* $\{\beta + n\alpha\}_{n \in \mathbb{Z}} \cap (\Delta \cup \{0\})$ *is a finite connected string* $\{\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta, \ldots, \beta + (q-1)\alpha, \beta + q\alpha\}$, *where* $p, q \in \mathbb{Z}_{\geq 0}$ *and* $p - q = \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$.

c) *If* $\alpha, \beta, \alpha + \beta \in \Delta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

d) *If* $\alpha \in \Delta$, *then* $n\alpha \in \Delta$ *if and only if* $n = 1$ *or* $n = -1$.

Proof. **a)** Suppose that $\dim \mathfrak{g}_\alpha > 1$ for some $\alpha \in \Delta$, then $\dim \mathfrak{g}_{-\alpha} > 1$ by non-degeneracy of the restriction $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$, property 3 above. Consider the adjoint representation of the subalgebra $\mathfrak{a}_\alpha = \mathbb{F}E + \mathbb{F}F + \mathbb{F}H$ on $\mathfrak{g}$. In particular, recall $H = \frac{2\nu^{-1}(\alpha)}{K(\alpha,\alpha)}$. Since $\dim \mathfrak{g}_{-\alpha} > 1$, there exists a non-zero vector $v \in \mathfrak{g}_{-\alpha}$ such that $K(E, v) = 0$. Hence, $(\textbf{ad } E)v = [E, v] = K(E, v)\nu^{-1}(\alpha){=}0$. But $(\textbf{ad } H)v = [H, v] = \frac{2[\nu^{-1}(\alpha),v]}{K(\alpha,\alpha)} = \frac{-2\alpha(\nu^{-1}(\alpha))v}{K(\alpha,\alpha)} = \frac{-2K(\alpha,\alpha)v}{K(\alpha,\alpha)} = -2v$. Hence, $\dim \mathfrak{g} = \infty$ by the Key Lemma, which yields a contradiction.

b) Let q be the largest integer such that $\beta + q\alpha \in \Delta \cup \{0\}$. Notice $q \geq 0$. Pick a non-zero vector $v \in \mathfrak{g}_{\beta+q\alpha}$. Then, $(\textbf{ad } E)v = 0$ since it lies in $\mathfrak{g}_{\beta+(q+1)\alpha}$. Also, $(\textbf{ad } H)v = (\beta + q\alpha)(H)v = \left(\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} + 2q\right) v$. Hence, by the Key Lemma: $\lambda := \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} + 2q \in \mathbb{Z}_{\geq 0}$ and $(\textbf{ad } F)^j v$ are non-zero vectors for $0 \leq j \leq \lambda$. But $(\textbf{ad } F)^j v \in \mathfrak{g}_{\beta+(q-j)\alpha}$, so $\beta+q\alpha, \beta+(q-1)\alpha, \ldots, \beta+(q-\lambda)\alpha \in \Delta \cup \{0\}$. Define $p := -(q - \lambda) = q + \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$. Let p' be the largest integer for which $\beta - p'\alpha \in \Delta \cup \{0\}$. Again, notice $p' \geq 0$. Pick a non-zero vector $v' \in \mathfrak{g}_{\beta-p'\alpha}$. Then, $(\textbf{ad } F)v' = 0$ and $(\textbf{ad } H)v' = \left(\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} - 2p'\right) v'$. By the corollary of the Key Lemma, we conclude that $-\lambda' := 2p' - \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} \in \mathbb{Z}_{\geq 0}$ and that $\beta - p'\alpha, \beta - (p' - 1)\alpha, \ldots, \beta - (p' + \lambda')\alpha \in \Delta \cup \{0\}$. Define $q' := -(p' + \lambda') = p' - \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$. Since q and p' are the largest integers for which $\beta + q\alpha \in \Delta \cup \{0\}$ (resp. $\beta - p'\alpha \in \Delta \cup \{0\}$), we conclude that $q \geq q'$ and $p' \geq p$. Hence, $\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} = p - q \leq p' - q' = \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$, showing that $p = p'$, $q = q'$, $p, q \in \mathbb{Z}_{\geq 0}$.

c) Pick the largest integers p and q such that $\beta - p\alpha, \beta + q\alpha \in \Delta \cup \{0\}$. Pick a non-zero vector $v \in \mathfrak{g}_{\beta-p\alpha}$. Then, $(\textbf{ad } F)v = 0$ and $(\textbf{ad } H)v = \left(\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} - 2p\right) v$. By the corollary of the Key Lemma, $(\textbf{ad } E)^j v \neq 0$ for $0 \leq j \leq 2p - \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} = p + q$. But $q \geq 1$ since $\alpha + \beta \in \Delta$, so $(\textbf{ad } E)^{p+1} v$ is a non-zero vector. Its corresponding root is $\alpha + \beta$, and $(\textbf{ad } E)^p v \in \mathfrak{g}_\beta$, so $[E, \mathfrak{g}_\beta] \neq 0$. Hence, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ since $\dim \mathfrak{g}_{\alpha+\beta} = 1$.

d) Let $\beta = n\alpha$, $n \neq 0$. Then, $\frac{2K(\alpha,\beta)}{K(\beta,\beta)} = \frac{2n}{n^2} = \frac{2}{n} \in \mathbb{Z}$. Hence, either $n = 2, 1, -1$ or -2. However, $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0$ by **a)** (resp. $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\alpha}] = 0$), so 2α (resp. -2α) is not a root because otherwise, **c)** would imply that $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$ (resp. $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{-2\alpha}$).

□

Exercise 13.3. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$. We know K is non-degenerate, so $\mathfrak{g}$ is semisimple. We will find all possibilities for p and q in the proof above. Suppose $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with associated root system Δ. Under an inner automorphism σ of $\mathfrak{g}$, the Cartan subalgebra $\mathfrak{h}$ is sent to a conjugate Cartan Subalgebra $\mathfrak{h}' := \sigma(\mathfrak{h})$, and Δ is sent to the root system Δ' consisting of all linear functionals on $\mathfrak{h}'$ of the form $\alpha\sigma^{-1}$ with $\alpha \in \Delta$. Hence, we have the root space decomposition $\mathfrak{g} = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha' \in \Delta'} \mathfrak{g}_{\alpha'} \right)$, where $\mathfrak{g}_{\alpha'} = \{a \in \mathfrak{g} | [h, a] = \alpha'(h)a \text{ for all } h \in \mathfrak{h}'\}$. However, inner automorphisms preserve the trace and we can check that $K(\alpha, \beta) = K(\alpha\sigma^{-1}, \beta\sigma^{-1})$, so the values of p and q are independent of the choice of Cartan subalgebra.

To construct $\mathfrak{h}$, take a diagonal matrix $a = \text{diag } (a_1, a_2, \ldots, a_n) \in \mathfrak{g}$ all of whose diagonal entries are distinct. By the extension of Exercise 3 in Lecture 7 to $\mathfrak{sl}_n(\mathbb{F})$, a is regular in $\mathfrak{g}$. Hence, $\mathfrak{g}_0^a$ is a Cartan subalgebra of $\mathfrak{g}$, so let $\mathfrak{h} = \mathfrak{g}_0^a$. As $(\textbf{ad } a)^N e_{i,j} = (a_i - a_j)^N e_{i,j}$ for all $N \geq 0$, we see that $\mathfrak{h}$ is precisely the set of diagonal matrices of $\mathfrak{g}$. A basis for $\mathfrak{h}^*$ is given by $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n\}$, where $\varepsilon_i(b) = b_i$ for any $b = \text{diag}\,(b_1, b_2, \ldots, b_n) \in \mathfrak{h}$ and $i \in \{1, \ldots, n\}$. We can check that:

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{F}e_{i,j} \text{ for all } i, j \in \{1, \ldots, n\},\ i \neq n.$$

Hence, the set $\Delta := \{\varepsilon_i - \varepsilon_j | i, j \in \{1, \ldots, n\},\ i \neq j\}$ is a root system for $\mathfrak{g}$. For no pair of roots $\alpha, \beta \in \Delta$ it is true that $\beta + 3\alpha \in \Delta \cup \{0\}$. Thus, the only possibilities for (q, p) are $(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)$.

When $n = 2$, we can only have $(q, p) = (2, 0), (0, 2)$. Let $\alpha := \varepsilon_1 - \varepsilon_2$. Then $\Delta = \{\pm\alpha\}$, so $\alpha = \alpha + (0)\alpha$, $0 = \alpha - (1)\alpha$, $-\alpha = \alpha - (2)\alpha$ and we have $\alpha, 0, -\alpha \in \Delta \cup \{0\}$, giving all possible values for p and q.

If $n = 3$, we can only have $(q, p) = (2, 0), (0, 2), (1, 0), (0, 1)$. We have the pairs $(2, 0)$ and $(0, 2)$ by the previous case. Now, letting $\beta := \varepsilon_1 - \varepsilon_3$ and $\gamma := \varepsilon_2 - \varepsilon_3$ so that $\Delta : \{\pm\alpha, \pm\beta, \pm\gamma\}$, we see that $\alpha - \beta, \alpha \in \Delta \cup \{0\}$ but $\alpha - 2\beta, \alpha + \beta \notin \Delta \cup \{0\}$, and similar relations hold among α, γ and β, γ by symmetry.

If $n \geq 4$, we can only have $(q, p) = (2, 0), (0, 2), (1, 0), (0, 1), (0, 0)$. The first four pairs come from the previous two cases. The fifth pair $(0, 0)$ occurs if we let $\delta = \varepsilon_3 - \varepsilon_4$ and then notice that $\alpha - \delta, \alpha + \delta \notin \Delta \cup \{0\}$.

In general, let $\alpha_{ij} := \varepsilon_i - \varepsilon_j$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then, for all multisets $\{i, j, k, l\} \subseteq \{1, \ldots, n\}$:

if $\{i, j\} \cap \{k, l\} = 2$, then α_{ij} and α_{kl} are related by pairs $(2, 0), (0, 2)$;

if $\{i, j\} \cap \{k, l\} = 1$, then α_{ij} and α_{kl} are related by pairs $(1, 0), (0, 1)$;

if $\{i, j\} \cap \{k, l\} = 0$, then α_{ij} and α_{kl} are related by the pair $(0, 0)$.