18.745 Introduction to Lie Algebras October 19, 2010

Lecture 12 — Structure Theory of Semisimple Lie Algebras (I)

Prof. Victor Kac *Scribes: Mario DeFranco and Christopher Policastro*

In lecture 10, we saw that Cartan's criterion holds without requiring the base field to be algebraically closed. Similarly, we can deduce the semisimplicity criterion proved in the last lecture assuming only that the base field has characteristic zero.

Exercise 12.1. Let $\mathfrak{g}$ denote a Lie algebra over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Show that $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is nondegenerate.

Solution: When we proved the theorem in lecture 11, we used the algebraic closure of the base field $\mathbb{F}$ only to apply Cartan's criterion. We know from exercise 10.4, however, that Cartan's criterion is valid assuming merely that $\mathbb{F}$ has characteristic 0. So the same proof applies in this case. □

We should not be led to think, however, that the assumption of algebraic closure is never necessary. Merely requiring that the base field has characteristic zero is not enough for instance to derive Chevalley's theorem on the conjugacy of Cartan subalgebras.

Exercise 12.2. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. Show that there are two distinct conjugacy classes of Cartan subalgebras given by

$$\mathfrak{h}_1 = \mathbb{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{and} \quad \mathfrak{h}_2 = \mathbb{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} .$$

Solution: Let $\{e, f, h\}$ be the standard basis of $\mathfrak{g}$ where $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. We want to show that there exist two distinct conjugacy classes of Cartan subalgebras in $\mathfrak{g}$ given by $\mathbb{R}h$ and $\mathbb{R}(e - f)$.

If $w = ae + bf + ch$, then the characteristic polynomial of $\mathbf{ad}\ w$ is

$$t^3 - 4(c^2 + ab)\ . \tag{1}$$

This fact follows from an immediate computation. So an element $w = ae + bf + ch$ is regular iff $c^2 + ab \neq 0$, and nilpotent otherwise. In particular h and $e - f$ are regular with respective eigenvalues $\{0, \pm 2\}$ and $\{0, \pm 2i\}$.

Suppose we have a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\overline{\mathfrak{g}} = \mathfrak{sl}(\mathbb{C})$ and $\overline{h} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{h}$. By exercise 10.2, we know that $\mathfrak{g}$ is not nilpotent, so $\mathfrak{h}$ has dimension 1 or 2. If $\mathfrak{h}$ has dimension 2, then by extending a basis of $\mathfrak{g}$ to a basis of $\overline{\mathfrak{g}}$, we see that $\overline{\mathfrak{h}}$ is nilpotent and self-normalizing. Yet the Cartan subalgebras of $\overline{\mathfrak{g}}$ are 1-dimensional. Therefore $\mathfrak{h}$ is 1-dimensional.

By exercise 10.4, if w is a regular element, then $\mathfrak{g}_0^w \supset \mathbb{R}w$ is a Cartan subalgebra. So $\mathfrak{g}_0^w = \mathbb{R}w$, implying $\mathbb{R}w$ is Cartan. Conversely, if $\mathbb{R}w$ is Cartan, then w is regular. Otherwise w would be nilpotent contradicting the fact that $\mathbb{R}w$ is self-normalizing. Therefore all Cartan subalgebras are of the form $\mathbb{R}w$ for a regular element w.

By (1), we see that w has eigenvalues $\left\{0, \pm(c^2 + ab)^{1/2}\right\}$. Since scaling determines a Lie algebra isomorphism, and $c^2 + ab \neq 0$, it follows that w is conjugate to an element of $\mathbb{R}h$ or $\mathbb{R}(e - f)$. But h is not conjugate to an element of $\mathbb{R}(e - f)$, since conjugate elements have the same eigenvalues.

□

In this lecture and the few that follow, we will study the structure of finite dimensional semisimple Lie algebras with the aim of classifying them. This will amount to a detailed knowledge of root space decompostions. The first step is to use the Killing form to understand Cartan subalgebras and their actions under the adjoint representation.

Unless otherwise stated, we will assume throughout that our base field $\mathbb{F}$ is algebraically closed of characteristic zero.

We begin by generalizing our notion of Jordan decomposition to an arbitrary Lie algebra.

Definition 12.1. An *abstract Jordan decomposition* of an element of a Lie algebra $\mathfrak{g}$ is a decomposition of the form $a = a_s + a_n$, where

(a) $\mathbf{ad}\ a_s$ is a diagonalizable (equivalently semisimple) endomorphism of $\mathfrak{g}$.

(b) $\mathbf{ad}\ a_n$ is a nilpotent endomorphism.

(c) $[a_s, a_n] = 0$.

Exercise 12.3. Abstract Jordan decomposition in a Lie algebra $\mathfrak{g}$ is unique when it exists if and only if $Z(\mathfrak{g}) = 0$.

Solution: ($\Rightarrow$) Suppose $c \in Z(\mathfrak{g})$ is not 0, and that $a \in \mathfrak{g}$ has abstract Jordan decomposition $a = a_s + a_n$. Let $a'_s = a_s - c$ and $a'_n = a_n + c$. We note the following facts: $a = (a_s - c) + (a_n + c) = a'_s + a'_n$; $[a'_s, a'_n] = [a_s, a_n] = 0$; $\mathbf{ad}\ a'_s = \mathbf{ad}\ a_s$ and $\mathbf{ad}\ a'_n = \mathbf{ad}\ a_n$ as $Z(\mathfrak{g}) = \ker \mathbf{ad}$. Since $\mathbf{ad}\ a_n$ is nilpotent, and $\mathbf{ad}\ a_s$ is semisimple, we conclude that $a = a'_s + a'_n$ is an abstract Jordan decomposition. But $a'_s \neq a_s$ as $c \neq 0$. So the decomposition for a is not unique.

($\Leftarrow$) Suppose that $a = a_s + a_n = a'_s + a'_n$ are abstract Jordan decompositions for some $a \in \mathfrak{g}$. Since $\mathbf{ad}\ (a_s + a_n)$ and $\mathbf{ad}\ (a'_s + a'_n)$ are commuting operators, and $\mathbf{ad}\ a = \mathbf{ad}\ a_s + \mathbf{ad}\ a_n$ and $\mathbf{ad}\ a = \mathbf{ad}\ a'_s + \mathbf{ad}\ a'_n$ are Jordan decompositions, we know from a lemma of lecture 6, that

$$[\mathbf{ad}\ a'_s, \mathbf{ad}\ a_s + \mathbf{ad}\ a_n] = 0, \quad [\mathbf{ad}\ a'_n, \mathbf{ad}\ a_s + \mathbf{ad}\ a_n] = 0$$

and consequently

$$[\mathbf{ad}\ a'_s, \mathbf{ad}\ a_s] = 0 \quad [\mathbf{ad}\ a'_n, \mathbf{ad}\ a_n] = 0\ .$$

Since the sum of commuting semisimple (resp. nilpotent) operators is semisimple (resp. nilpotent),

$$\mathbf{ad}\ a'_s - \mathbf{ad}\ a_s = \mathbf{ad}\ a_n - \mathbf{ad}\ a'_n \;\Rightarrow\; \mathbf{ad}\ a'_s - \mathbf{ad}\ a_s = 0,\ \mathbf{ad}\ a_n - \mathbf{ad}\ a'_n = 0\ .$$

As $\ker \mathbf{ad} = Z(\mathfrak{g}) = 0$ we conclude that $a_n = a'_n$ and $a_s = a'_s$.

□

Our first result in this lecture will be to show the existence of abstract Jordan decompositions under the assumption of semisimplicity.

Theorem 12.1. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{F}$.*

(a) $Z(\mathfrak{g}) = 0$.

(b) All derivations of $\mathfrak{g}$ are inner.

(c) Any $a \in \mathfrak{g}$ admits a unique Jordan decomposition.

Proof. (a): One of our equivalent definitions of semisimplicity is that $\mathfrak{g}$ has no nontrivial abelian ideals. Hence $Z(\mathfrak{g}) = 0$.

(b): By construction $\mathfrak{g} \xrightarrow{\mathbf{ad}} \mathrm{Der}\ \mathfrak{g}$ with kernel $Z(\mathfrak{g})$. Since $Z(\mathfrak{g}) = 0$, this implies $\mathfrak{g} \cong \mathbf{ad}\ \mathfrak{g}$.

Consider the trace form on Der $\mathfrak{g}$ under the tautological representation. The restriction to $\mathbf{ad}\ \mathfrak{g}$ is the Killing form on $\mathfrak{g}$, which is nondegenerate by semsimplicity. From a lemma of lecture 11, we have that $\mathrm{Der}\ \mathfrak{g} = \mathbf{ad}\ \mathfrak{g} \oplus \mathbf{ad}\ \mathfrak{g}^{\perp}$ as a direct sum of Lie algebras.

We need to show that $\mathbf{ad}\ \mathfrak{g}^{\perp} = 0$. In the contrary case, there exists a nonzero element $D \in \mathbf{ad}\ \mathfrak{g}^{\perp}$. For $a \in \mathfrak{g}$, we have $[D, \mathbf{ad}\ a] = 0$. We recall from exercise 2.1 that $[D, \mathbf{ad}\ a] = \mathbf{ad}\ D(a)$. Hence $D(a) \in Z(\mathfrak{g})$ for any $a \in \mathfrak{g}$. Since $Z(\mathfrak{g}) = 0$, we conclude that $D = 0$ in contradiciton to our assumption.

(c): Fix $a \in \mathfrak{g}$. Consider the classical Jordan decompostion of $\mathbf{ad}\ a = A_s + A_n$, where A_s is diagonalizable, A_n is nilpotent, and $[A_s, A_n] = 0$.

Consider the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to $\mathbf{ad}\ a$

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_{\lambda}^{a} .$$

Recall that $\mathfrak{g}_{\lambda}^{a} = \left\{(\mathbf{ad}\ a - \lambda I)^N = 0\right\}$, and $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ with $A_s \mid_{\mathfrak{g}_\lambda} = \lambda I$.

We want first to show that A_s is a derivation. By linearity, it is enough to check this for $x \in \mathfrak{g}_{\lambda}^{a}$ and $y \in \mathfrak{g}_{\mu}^{a}$. We have

$$A_s([x, y]) = (\lambda + \mu) \cdot [x, y] = [\lambda x, y] + [x, \mu y] = [A_s x, y] + [x, A_s y] .$$

This means that $A_s \in \mathrm{Der}\ \mathfrak{g}$.

By part (b), all derivations are inner. So there exists $a_s \in \mathfrak{g}$ such that $\mathbf{ad}\ a_s = A_s$. Letting $a_n = a - a_s$, we have that $\mathbf{ad}\ a_n = A_n$. It remains to check that $[a_s, a_n] = 0$. Since

$$\mathbf{ad}\ ([a_s, a_n]) = [\mathbf{ad}\ a_s, \mathbf{ad}\ a_n] = [A_s, A_n] = 0$$

part (a) gives the result.

□

With the previous result under our belts, let us get a firmer understanding of the Cartan subalgebras of $\mathfrak{g}$ from Theorem 12.1.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and consider the generalized root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \quad \text{with} \quad \mathfrak{g}_0 = \mathfrak{h}$$

where $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Recall that in the case of the adjoint representation, we say generalized root space decomposition instead of generalized weight space decomposition. It is important to make this distinction; we will see its convenience in later lectures, as we try to better understand the functionals α appearing in the decomposition.

Theorem 12.2. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be as defined, and K denote the Killing form on $\mathfrak{g}$.*

(a) $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *if* $\alpha + \beta \neq 0$.

(b) $K\,|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ *is nondegenerate. Consequently* $K\,|_{\mathfrak{h}\times\mathfrak{h}}$ *is nondegenerate.*

(c) $\mathfrak{h}$ *is an abelian subalgebra of* $\mathfrak{g}$.

(d) $\mathfrak{h}$ *consists of semisimple elements; namely, each* $\mathbf{ad}\ h$ *for* $h \in \mathfrak{h}$ *is diagonalizable. Consequently* $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \ \ \forall h \in \mathfrak{h}\}$.

Proof. (a): Let $a \in \mathfrak{g}_\alpha$ and $b \in \mathfrak{g}_\beta$ with $\alpha + \beta \neq 0$. Note that

$$((\mathbf{ad}\ a)(\mathbf{ad}\ b))^N \mathfrak{g}_\gamma \subset \mathfrak{g}_{\gamma + N\alpha + N\beta} = 0$$

for $N \gg 0$ as $\mathfrak{g}$ is finite dimensional. So we can choose N sufficiently large so that the operator $((\mathbf{ad}\ a)(\mathbf{ad}\ b))^N$ is zero on each summand of the decomposition. Therefore $(\mathbf{ad}\ a)(\mathbf{ad}\ b)$ is nilpotent. By exercise 3.4, the eigenvalues of $(\mathbf{ad}\ a)(\mathbf{ad}\ b)$ are zero implying that

$$K(a, b) = \mathrm{tr}\ _{\mathfrak{g}}((\mathbf{ad}\ a)(\mathbf{ad}\ b)) = 0\ .$$

(b): By semisimplicity, we know that K is nondegenerate on $\mathfrak{g}$. By part (a), we have $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha \neq -\beta$. So necessarily $K\,|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is nondegenerate.

(c): Consider $\mathbf{ad}\ \mathfrak{h}$, and note that $K\,|_{\mathfrak{h}\times\mathfrak{h}}$ is the trace form on $\mathbf{ad}\ \mathfrak{h}$ under the tautological representation. As $\mathfrak{h}$ is solvable, indeed nilpotent, we know that $\mathbf{ad}\ \mathfrak{h}$ is solvable. So by Cartan's criterion, we find that

$$0 = \mathrm{tr}\ _{\mathfrak{g}}(\mathbf{ad}\ \mathfrak{h}, [\mathbf{ad}\ \mathfrak{h}, \mathbf{ad}\ \mathfrak{h}]) = K(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}])\ .$$

By part (b), however, $K\,|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. Therefore we conclude that the derived subalgebra $[\mathfrak{h}, \mathfrak{h}]$ is zero.

(d): Consider $h \in \mathfrak{h}$. By Theorem 12.1 (c), h has an abstract Jordan decomposition of the form $h = h_s + h_n$, where $h_s, h_n \in \mathfrak{g}$ are such that $\mathbf{ad}\ h_s$ is diagonalizable, $\mathbf{ad}\ h_n$ is nilpotent, and $[h_s, h_n] = 0$.

By part (c), $[h, \mathfrak{h}] = 0$. Hence for $h' \in \mathfrak{h}$, we have $0 = \mathbf{ad}\ ([h', h]) = [\mathbf{ad}\ h', \mathbf{ad}\ h]$. So by a lemma from lecture 6, we know that $[\mathbf{ad}\ h', (\mathbf{ad}\ h)_s] = 0$, yielding

$$0 = [\mathbf{ad}\ h', (\mathbf{ad}\ h)_s] = [\mathbf{ad}\ h', \mathbf{ad}\ h_s] = \mathbf{ad}\ ([h', h_s])\ .$$

By Theorem 12.1 (a), this implies $[h', h_s] = 0$ for $h' \in \mathfrak{h}$. But since $\mathfrak{h}$ is a maximal nilpotent subalgebra, we conclude that $h_s \in \mathfrak{h}$.

It remains to be shown that $h_n = 0$. Note that $h_n = h - h_s \in \mathfrak{h}$.

Since $\mathbf{ad}\ \mathfrak{h}$ is solvable, Lie's theorem implies that there exists a basis of $\mathfrak{gl}_{\mathfrak{g}}$ such that the elements of $\mathbf{ad}\ \mathfrak{h}$ are upper triangular. As $\mathbf{ad}\ h_n$ is nilpotent, it is strictly upper triangular. Therefore $\mathrm{tr}\ ((\mathbf{ad}\ h')(\mathbf{ad}\ h_n)) = 0$ for $h' \in \mathfrak{h}$; namely $K(\mathfrak{h}, h_n) = 0$. By part (b), $K\,|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. So $h_n = 0$, implying that $h = h_s$ is diagonalizable. □

For arbitrary Lie algebras, Cartan subalgebras can have an unwieldy structure, and an unpredictable action under the adjoint representation. We see from the previous result, however, that with the assumption of semisimplicity the situation is much more transparent.

We can rewrite the root space decomposition as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where Δ is the set of all $\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq 0$. Recall that $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \ \ \forall h \in \mathfrak{h}\}$.

Definition 12.2. We call $\alpha \in \Delta$ a *root* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ the corresponding *root space*.

The rest of the classification will be concerned more or less with gathering information about roots of $\mathfrak{g}$. Having put the Killing form to good use in determining the structure of Cartan subalgebras, we would like to extend it to Δ. We have a canonical linear map

$$\nu : \mathfrak{h} \ni h \mapsto K(h, \bullet) \in \mathfrak{h}^* \ .$$

Since $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate by Theorem 12.2 (b), this implies ν is injective. As $\mathfrak{h}$ and $\mathfrak{h}^*$ have the same dimension, ν determines a vector space isomorphism.

Definition 12.3. Abusing notation, we define a bilinear form $K : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{F}$ by the rule $K(\gamma, \mu) = K(\nu^{-1}(\gamma), \nu^{-1}(\mu))$. Note that $K(\nu(h), \nu(h')) = \nu(h)(h') = \nu(h')(h)$.

Theorem 12.3. *(a) If $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, then $[e, f] = K(e, f)\nu^{-1}(\alpha)$.*

(b) If $\alpha \in \Delta$, then $K(\alpha, \alpha) \neq 0$.

Proof. (a): We know that $[e, f] \in \mathfrak{h}$. By Theorem 12.2 (b), $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. So it is enough to show that $K([e, f] - K(e, f)\nu^{-1}(\alpha), h') = 0$ for any $h' \in \mathfrak{h}$. We have

$$\begin{aligned} K([e, f] - K(e, f)\nu^{-1}(\alpha), h') &= K([e, f], h') - K(e, f)\nu(\nu^{-1}(\alpha))(h') \\ &= -K(e, [h', f]) - K(e, f)\alpha(h') \\ &= \alpha(h')K(e, f) - K(e, f)\alpha(h') = 0 \ . \end{aligned}$$

Note that we have used the invariance of K, and the fact that $h' \in \mathfrak{h}$, $f \in \mathfrak{g}_{-\alpha}$. This gives the result.

(b): By Theorem 12.2 (b), $K\mid_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is nondegenerate. So we can find $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ such that $K(e, f) = 1$. By part (a), we know that $[e, f] = \nu^{-1}(\alpha)$. Hence

$$[\nu^{-1}(\alpha), e] = \alpha(\nu^{-1}(\alpha))e = K(\alpha, \alpha)e$$

and similarly

$$[\nu^{-1}(\alpha), f] = \alpha(\nu^{-1}(\alpha))f = K(\alpha, \alpha)f \ .$$

Suppose to the contrary that $K(\alpha, \alpha) = 0$. By the above relations, we obtain a Lie algebra

$$\mathfrak{a} = \mathbb{F}e + \mathbb{F}f + \mathbb{F}\nu^{-1}(\alpha)$$

isomorphic to Heis_3. Recall that Heis_3 is solvable with center $[\mathrm{Heis}_3, \mathrm{Heis}_3]$.

So applying Lie's theorem to the restriction of the adjoint representation to $\mathfrak{a}$, we can find a basis of $\mathfrak{gl}_{\mathfrak{g}}$ such that $\nu^{-1}(\alpha) = [\mathfrak{a}, \mathfrak{a}]$ is strictly upper triangular. But by Theorem 12.2 (d), we know that $\nu^{-1}(\alpha) \in \mathfrak{h}$ is diagonalizable. So we conclude that $\nu^{-1}(\alpha) = 0$, and $\alpha = 0$. This contradicts definition 12.2, and the fact that $\alpha \in \Delta$.

□

Exercise 12.4. (a) Show that all derivations of the 2-dimensional nonabelian Lie algebra are inner. (b) Find Der Heis_3. Note that not all derivations are inner.

Solution (a) Let $\mathfrak{g}$ be the 2-dimensional nonabelian Lie algebra over $\mathbb{F}$ with ordered basis $\{e, f\}$ where $[e, f] = f$. Choose $D \in \text{Der}\ (\mathfrak{g})$ with

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } a, b, c, d \in \mathbb{F}.$$

We have

$$\begin{aligned} D([e, f]) &= [De, f] + [e, Df] \iff \\ D(f) &= [ae + cf, f] + [e, be + df] \iff \\ be + df &= (a + d)f \iff a = b = 0\ . \end{aligned}$$

Thus $D = \mathbf{ad}\ de - cf$ implying D is inner.

(b) Let $\mathfrak{g} = \text{Heis}_3$ over $\mathbb{F}$ with ordered basis $\{p, q, c\}$ where $[p, q] = c$. Choose $D \in \text{Der}\ (\text{Heis}_3)$ with $D = \sum_{1 \le i,j \le 3} a_{ij} \cdot e_{ij}$. Note that

$$[D, \mathbf{ad}\ p] = \begin{pmatrix} 0 & a_{13} & 0 \\ 0 & a_{23} & 0 \\ -a_{21} & a_{33} - a_{22} & -a_{23} \end{pmatrix}, \quad [D, \mathbf{ad}\ q] = \begin{pmatrix} -a_{13} & 0 & 0 \\ -a_{23} & 0 & 0 \\ -a_{33} + a_{11} & a_{12} & a_{13} \end{pmatrix}.$$

Recall from exercise 2.1 that $[D, \mathbf{ad}\ p] = \mathbf{ad}\ D(p)$ and $[D, \mathbf{ad}\ q] = \mathbf{ad}\ D(q)$. Since $\mathbf{ad}\ p = e_{32}$ and $\mathbf{ad}\ q = -e_{31}$, it follows that $a_{13} = a_{23} = 0$ and $a_{33} = a_{11} + a_{22}$; namely

$$D = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} \end{pmatrix}.$$

By checking on basis elements, we see that every matrix of this form is indeed a derivation.

Hence the derivations of $\mathfrak{g}$ form a 6-dimensional subspace, while the inner derivations form a 2-dimensional subspace.

□

Exercise 12.5. Show that Theorem 12.1 parts (b) and (c), and Theorem 12.2 part (c) hold for $\text{char}(\mathbb{F}) = 0$.

Solution (12.1 (b)): Note that Theorem 12.1 (a) makes no assumptions on $\mathbb{F}$. We check that the lemma from lecture 11, does not require that the $\mathbb{F}$ be algebraically closed. Hence we only use algebraic closure to deduce that the Killing form on $\mathfrak{g}$ is nondegenerate. This follows from exercise 12.1 merely assuming that $\mathbb{F}$ has characteristic zero.

(12.1(c)): Let $\overline{\mathfrak{g}} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{g}$. Choose $a \in \mathfrak{g}$. We know that there exist $a_s, a_n \in \overline{\mathfrak{g}}$ such that $a = a_s + a_n$ is an abstract Jordan decomposition. Choose a basis $\{g_1, \ldots, g_n\}$ of $\mathfrak{g}$, and note that $\{\overline{g}_1, \ldots, \overline{g}_n\} = \{1 \otimes g_1, \ldots, 1 \otimes g_n\}$ is a basis of $\overline{\mathfrak{g}}$.

Claim. $a_s \in \mathfrak{g}$:

Since $\mathrm{char}(\mathbb{F}) = 0$, this implies that $\overline{\mathbb{F}}/\mathbb{F}$ is a Galois extension. Recall that $f \in \mathbb{F}$ iff $\sigma(f) = f$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

Let $a_s = \sum_{1 \leq i \leq n} \alpha_i \overline{g}_i$. Suppose to the contrary that $\alpha_j \in \overline{\mathbb{F}} - \mathbb{F}$. There exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ such that $\sigma(\alpha_j) \neq \alpha_j$. Define a map

$$\Phi : \overline{\mathfrak{g}} \ni \sum_{1 \leq i \leq n} \beta_i \overline{g}_i \mapsto \sum_{1 \leq i \leq n} \sigma(\beta_i) \overline{g}_i \in \overline{\mathfrak{g}} \, .$$

Note that Φ is an additive bijection, and $\Phi \mid_{\mathfrak{g}}$ is the identity. Moreover

$$\Phi([\overline{g}_i, \overline{g}_j]) = [\overline{g}_i, \overline{g}_j] = [\Phi\overline{g}_i, \Phi\overline{g}_j] \Longrightarrow \Phi([g, g']) = [\Phi g, \Phi g'] \ \ \forall g, g' \in \overline{\mathfrak{g}}$$

since $[\overline{g}_i, \overline{g}_j] = [g_i, g_j] \in \mathfrak{g}$.

Hence we check that $a = \Phi a_s + \Phi a_n$ is an abstract Jordan decomposition of a. By exercise 12.3, we know that $\Phi a_s = a_s$. This is a contradiction.

□

Consequently $a_n \in \mathfrak{g}$. So $a = a_s + a_n$ is an abstract Jordan decomposition of a in $\mathfrak{g}$.

(12.2(c)): Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Choose $a \in \mathfrak{h}$ nonzero. By the previous result, we know that a has an abstract Jordan decompostion $a = a_s + a_n$. Hence we have a generalized eigenspace decomposition $\mathfrak{g} = \oplus_\alpha \mathfrak{g}^a_\alpha$, where $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Noting that the Killing form on $\mathfrak{g}$ is nondegenerate by exercise 12.1, the proofs of Theorem 12.2 (a) and (b) are valid replacing root spaces by generalized eigenspaces. So $K \mid_{\mathfrak{g}^a_0 \times \mathfrak{g}^a_0}$ is nondegenerate.

Note that $\mathfrak{h} \subset \mathfrak{g}^a_0$ since $\mathfrak{h}$ is nilpotent. Extending an orthogonal basis of $\mathfrak{h}$ to an orthogonal basis of $\mathfrak{g}^a_0$,we see that

$$K \mid_{\mathfrak{h} \times \mathfrak{h}} \text{ degenerate } \Longrightarrow K \mid_{\mathfrak{g}^a_0 \times \mathfrak{g}^a_0} \text{ degenerate } .$$

Therefore $K \mid_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate.

By exercise 10.4, we know that Cartan's criterion applies to $\mathfrak{g}$. Therefore the argument in Theorem 12.2 (c) is valid in this case.

□