18.745 Introduction to Lie Algebras	October 14, 2010

Lecture 11 — The Radical and Semisimple Lie Algebras

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Exercise 11.1. *Let $\mathfrak{g}$ be a Lie algebra. Then*

1. *if $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are ideals, then $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are ideals, and if $\mathfrak{a}$ and $\mathfrak{b}$ are solvable then $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are solvable.*

2. *If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal and $b \subset \mathfrak{g}$ is a subalgebra, then $\mathfrak{a} + \mathfrak{b}$ is a subalgebra*

Solution:

Let $x \in \mathfrak{g}, a \in \mathfrak{a}, b \in \mathfrak{b}$. Then $[x, a+b] = [x, a] + [x, b] \in \mathfrak{a} + \mathfrak{b}$ since $\mathfrak{a}, \mathfrak{b}$ are ideals. Similarly, if $y \in \mathfrak{a} \cap \mathfrak{b}$ then $[x, y] \in \mathfrak{a}$, $[x, y] \in \mathfrak{b}$, so $\mathfrak{a} \cap \mathfrak{b}$ is an ideal too.

By an easy induction, $(\mathfrak{a} \cap \mathfrak{b})^{(i)} \subset \mathfrak{a}^{(i)}$, which is eventually zero if $\mathfrak{a}$ is solvable, so $\mathfrak{a} \cap \mathfrak{b}$ is solvable.

We showed earlier (in lecture 4) that if $\mathfrak{h} \subset \mathfrak{g}$ and both $\mathfrak{h}, \mathfrak{g}/\mathfrak{h}$ are solvable, then $\mathfrak{g}$ is solvable. Consider $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$. By hypothesis, $\mathfrak{b}$ is solvable. Noether's Third Isomorphism Theorem shows $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$ (more explicitly, look at the homomorphisms $\mathfrak{a} \to \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} + \mathfrak{b} \to (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$, notice that the kernel of their composition is exactly $\mathfrak{a} \cap \mathfrak{b}$). Since $\mathfrak{a}$ is solvable, $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$ is solvable. Hence $\mathfrak{a} + \mathfrak{b}$ is solvable.

For part 2, given any $a_1, a_2 \in \mathfrak{a}$ and $b_1, b_2 \in \mathfrak{b}$, $[a_1 + b_1, a_2 + b_2] = [a_1, a_2] + [b_1, a_2] + [a_1, b_2] + [b_1, b_2]$. Notice the sum of the first three terms is in $\mathfrak{a}$ since it is an ideal, and the last term is in $\mathfrak{b}$ since it is a subalgebra, thus the entire sum is in $\mathfrak{a} + \mathfrak{b}$. The closure of $\mathfrak{a} + \mathfrak{b}$ under addition and scalar multiplication is obvious. Thus the statement is proven.

Definition 11.1. *A radical $R(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is a solvable ideal of $\mathfrak{g}$ of maximal possible dimension.*

Proposition 11.1. *The radical of $\mathfrak{g}$ contains any solvable ideal of $\mathfrak{g}$ and is unique.*

Proof. If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$, then $\mathfrak{a} + R(\mathfrak{g})$ is again a solvable ideal. Since $R(\mathfrak{g})$ is of maximal dimension, $\mathfrak{a} + R(\mathfrak{g}) = R(\mathfrak{g})$ and $\mathfrak{a} \subset R(\mathfrak{g})$. For uniqueness, if there are two distinct maximal dimensional solvable ideals of $\mathfrak{g}$, then by above explanation, the sum is actually equal to both ideals. Thus we have a contradiction. □

If $\mathfrak{g}$ is a finite dimensional solvable Lie algebra, then $R(\mathfrak{g}) = \mathfrak{g}$. The opposite case is when $R(\mathfrak{g}) = 0$.

Definition 11.2. *A finite dimensional Lie algebra $\mathfrak{g}$ is called semisimple if $R(\mathfrak{g}) = 0$.*

Proposition 11.2. *A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if either of the following two conditions holds:*

1. *Any solvable ideal of $\mathfrak{g}$ is 0.*
2. *Any abelian ideal of $\mathfrak{g}$ is 0.*

Proof. The first condition is obviously equivalent to semisimplicity.

Suppose that $\mathfrak{g}$ contains a non-zero solvable ideal $\mathfrak{r}$. For some k, we have

$$\mathfrak{r} \supset \mathfrak{r}^{(1)} \supset \mathfrak{r}^{(2)} \supset \cdots \supsetneq \mathfrak{r}^{(k)} = 0,$$

hence $\mathfrak{r}^{(k-1)}$ is a nonzero abelian ideal, since all the $\mathfrak{r}^{(i)}$ are ideals of $\mathfrak{g}$. Abelian ideals are solvable, so the other direction is obvious. □

Remark 11.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $R(\mathfrak{g})$ its radical. Then $\mathfrak{g}/R(\mathfrak{g})$ is a semisimple Lie algebra. Indeed, if $\mathfrak{g}/R(\mathfrak{g})$ contains a non-zero solvable ideal $\mathfrak{f}$, then its preimage $\mathfrak{r}$ contains $R(\mathfrak{g})$ properly, so that $\mathfrak{r}/R(\mathfrak{g}) \cong \mathfrak{f}$, which is solvable, hence $\mathfrak{r}$ is a larger solvable ideal than $R(\mathfrak{g})$, a contradiction.*

So an arbitary finite dimensional Lie algebra "reduces" to a solvable Lie algebra $R(\mathfrak{g})$ and a semisimple Lie algebra $\mathfrak{g}/R(\mathfrak{g})$.

In the case char $\mathbb{F} = 0$ a stronger result holds:

Theorem 11.1. *(Levi decomposition) If $\mathfrak{g}$ is a finite dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0, then there exists a semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$, complementary to the radical $R(\mathfrak{g})$, such that $\mathfrak{g} = \mathfrak{s} \oplus R(\mathfrak{g})$ as vector spaces.*

Definition 11.3. *A decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ (direct sum of vector spaces), where $\mathfrak{h}$ is a subalgebra and $\mathfrak{r}$ is an ideal is called a semi-direct sum of $\mathfrak{h}$ and $\mathfrak{r}$ and is denoted by $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}$. The special case when $\mathfrak{h}$ is an ideal as well corresponds to the direct sum: $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r} = \mathfrak{h} \oplus \mathfrak{r}$.*

Remark 11.2. *The open end of $\ltimes$ goes on the side of the ideal. When both are ideals, we use $\times$ or $\oplus$, and the sum is direct.*

Exercise 11.2. *Let $\mathfrak{h}$ and $\mathfrak{r}$ be Lie algebras and let $\gamma : \mathfrak{h} \to \mathrm{Der}(\mathfrak{r})$ be a Lie algebra homomorphism. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ be the direct sum of vector spaces and extend the bracket on $\mathfrak{h}$ and on $\mathfrak{r}$ to the whole of $\mathfrak{g}$ by letting*

$$[h, r] = -[r, h] = \gamma(h)(r)$$

for $h \in \mathfrak{h}$ and $r \in \mathfrak{r}$. Show that this provides $\mathfrak{g}$ with a Lie algebra structure, $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}$, and that any semidirect sum of $\mathfrak{h}$ and $\mathfrak{r}$ is obtained in this way. Finally, show that $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r}$ if and only if $\gamma = 0$.

Solution:

The bracket so defined is clearly skew-symmetric. Restricted to $\mathfrak{h}$ or $\mathfrak{r}$ it satisfies the Jacobi identity. Take $h_1, h_2 \in \mathfrak{h}$ and $r_1, r_2 \in \mathfrak{r}$. Then we have $[h_1, [r_1, r_2]] + [r_1, [r_2, h_1]] + [r_2, [h_1, r_1]] = \gamma(h_1)([r_1, r_2]) - [r_1, \gamma(h_1)(r_2)] + [r_2, \gamma(h_1)(r_1)] = [\gamma(h_1)(r_1), r_2] + [r_1, \gamma(h_1)(r_2)] - [r_1, \gamma(h_1)(r_2)] + [r_2, \gamma(h_1)(r_1)] = 0$. We used the fact that $\gamma(h_1)$ is a derivation.

Furthermore, $[h_1, [h_2, r_1]] + [h_2, [r_1, h_1]] + [r_1, [h_1, h_2]] = [h_1, \gamma(h_2)(r_1)] - [h_2, \gamma(h_1)(r_1)] - \gamma([h_1, h_2])(r_1) = \gamma(h_1)\gamma(h_2)(r_1) - \gamma(h_2)\gamma(h_1)(r_1) - (\gamma(h_1)\gamma(h_2)(r_1) - \gamma(h_2)\gamma(h_1)(r_1)) = 0$. We used the fact that $\gamma : \mathfrak{h} \to \mathrm{Der}(\mathfrak{r})$ is a Lie algebra homomorphism. It follows that $[,]$ is a Lie bracket. Clearly $\mathfrak{r}$ is an ideal under the bracket, so $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}$.

Conversely, by the Jacobi identity, any bracket on $\mathfrak{h} \ltimes \mathfrak{r}$ is a homomorphism from $\mathfrak{h}$ to $\mathrm{Der}(\mathfrak{r})$ (essentially we just reverse the two calculations above). Finally, if $\gamma = 0$ then $[h, r] = 0$ for all $h \in \mathfrak{h}, r \in \mathfrak{r}$, and if $[h, r] = 0$ for all h, r then plainly $\gamma(h) = 0$ for all h, so $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r}$ if and only if $\gamma = 0$.

Exercise 11.3. *Let $\mathfrak{g} \subset gl_n(\mathbb{F})$ be a subspace consisting of matrices with arbitrary first m rows and 0 for the rest of the rows. Find $R(\mathfrak{g})$ and a Levi decomposition of $\mathfrak{g}$.*

Solution:

Write $x \in \mathfrak{g}$ as (A, B), where A is the upper-left $m \times m$ matrix block and B is the upper-right $m \times (n-m)$ block of x. Take $y = (A', B')$. Then it is clear that $[x, y] = ([A, A'], AB' - A'B)$. Hence, if $\mathfrak{h} \subset gl_m \hookrightarrow \mathfrak{g}$ is an ideal of $\mathfrak{gl}_m$, then $(\mathfrak{h}, 0) + R$ is an ideal of $\mathfrak{g}$, where R denotes the set of all (A, B) with $A = 0$. Furthermore, $\mathfrak{h}$ is solvable if and only if $\mathfrak{h} + R$ is solvable, because of $[R, R] = 0$ and the above identity. Notice that the radical of $\mathfrak{g}$ obviously contains R. Hence, the radical of $\mathfrak{g}$ corresponds to the radical of $\mathfrak{gl}_m$, i.e. $R(\mathfrak{g}) = (R(\mathfrak{gl}_m), 0) + R$. But, by the notes and problem 4 below, the radical of gl_m is $\mathbb{F}I$, the scalar matrices. Hence $R(\mathfrak{g}) = (\mathbb{F}I, 0) + R$ (sum of ideals). The complement of this can obviously be the subalgebra $(\mathfrak{sl}_m, 0)$.

Theorem 11.2. *Let V be a finite-dimensional vector space over an algebraically closed field of characteristic 0 and let $\mathfrak{g} \subset \mathfrak{gl}_V$ be a subalgebra, which is irreducible i.e. any subspace $U \subset V$, which is $\mathfrak{g}-$invariant, is either 0 or V. Then one of two possibilities hold:*

1. *$\mathfrak{g}$ is semisimple*
2. *$\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{sl}_V) \oplus \mathbb{F}I$ and $\mathfrak{g} \cap \mathfrak{sl}_V$ is semisimple.*

Proof. If $\mathfrak{g}$ is not semisimple, then $R(\mathfrak{g})$ is a non-zero solvable ideal in $\mathfrak{g}$. By Lie's theorem, there exists $\lambda \in R(\mathfrak{g})^*$ such that $V_\lambda = \{v \in V | av = \lambda(a)v, a \in R(\mathfrak{g})\}$ is nonzero. By Lie's lemma, V_λ is invariant. Hence, by irreducibility $V_\lambda = V$. Hence $a = \lambda(a)I_V$ for all $a \in R(\mathfrak{g})$, so $R(\mathfrak{g}) = \mathbb{F}I$. Hence $(\mathfrak{g} \cap \mathfrak{sl}_V) \cap R(\mathfrak{g}) = 0$, which proves that we have case 2, as $\mathfrak{g} \cap \mathfrak{sl}_V$ is semisimple since it is the complement of the radical.

□

Exercise 11.4. *Let V be finite-dimensional over a field $\mathbb{F}$ which is algebraically closed and characteristic 0. Show that $\mathfrak{gl}_V$ and $\mathfrak{sl}_V$ are irreducible subalgebras of $\mathfrak{gl}_V$. Deduce that $\mathfrak{sl}_V$ is semisimple.*

Solution: Suppose $W \subset V, W \neq 0$ is fixed by $\mathfrak{sl}_n$. Take nonzero vector $w = \sum c_i v_i \in W$, where $\{v_i\}$ is a basis of V. Suppose $c_k \neq 0$, and pick $\ell \neq k$. Then $e_{\ell k} \in \mathfrak{sl}_n$, and $e_{\ell k} w = c_k e_\ell \in W$. For every $m \neq \ell, e_{m\ell} \in \mathfrak{sl}_n$, so $e_{m\ell} e_\ell = e_m \in W$. Hence $W = V$. Since $\mathfrak{sl}_n \subset \mathfrak{gl}_n$, $\mathfrak{gl}_n$ is also irreducible. It follows from the theorem that $\mathfrak{sl}_n \cap \mathfrak{sl}_n = \mathfrak{sl}_n$ is semisimple.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Recall the Killing form on $\mathfrak{g}$: $K(a, b) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\ a)(\mathrm{ad}\ b)$.

Theorem 11.3. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Then the Killing form on $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is semisimple. Moreover, if $\mathfrak{g}$ is semisimple and $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then the restriction of the Killing form to $\mathfrak{a}, K|_{\mathfrak{a}\times\mathfrak{a}}$, is also non-degenerate and coincides with the Killing form of $\mathfrak{a}$.*

Exercise 11.5. *Let V be a finite-dimensional vector space with a symmetric bilinear form $(,)$. Let U be a subspace such that the restriction $(,)|_{U\times U}$ is non-degenerate. Denote $U^\perp = \{v \in V | (v, U) = 0\}$. Then $V = U \oplus U^\perp$.*

Solution: We pick an arbitrary basis $u_1, ..., u_m$ of U, and then extend it to a basis of V: $u_1, ..., u_m, ..., u_n$. Let the matrix associated with the given bilinear form relative to this basis be $Q = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$, where A is an $m \times m$ invertible matrix. We want to change the basis (more specifically, the part $u_{m+1}, ..., u_n$ to $u'_{m+1}, ..., u'_n$) to make part B vanish. Suppose that the base change matrix is $P = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$, where the sizes of the blocks match that of Q. Then the new matrix associated to the bilinear form is $P^T Q P = \begin{pmatrix} 1 & 0 \\ X^T & 1 \end{pmatrix} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & AX + B \\ X^T A + B^T & * \end{pmatrix}$. Thus we can just take $X = -A^{-1}B$ and the new matrix will have zero upper right block. It is obvious now that we have the desired decomposition by noticing that $U^\perp$ equals span of $u_{m+1}, ..., u_n$.

Lemma 11.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $(,)$ be a symmetric invariant bilinear form on $\mathfrak{g}$. Then*

1. *If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{a}^\perp$ is also an ideal.*
2. *If $(,)|_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate, then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, a direct sum of Lie algebras.*

Proof. 1. $v \in \mathfrak{a}^\perp$ means $(v, \mathfrak{a}) = 0$. If $b \in \mathfrak{g}$, then $([v, b], \mathfrak{a}) = (v, [b, \mathfrak{a}]) = 0$, since the form is invariant and $\mathfrak{a}$ is an ideal. Hence $\mathfrak{a}^\perp$ is an ideal.

2. Follows from the preceeding exercise and part 1.

$\square$

Proof of the theorem. Suppose K is non-degenerate on $\mathfrak{g}$, but $\mathfrak{g}$ is not semisimple. Hence there exists an abelian ideal $\mathfrak{a} \subset \mathfrak{g}$. But then $K(\mathfrak{a}, \mathfrak{g}) = 0$, contradicting non-degeneracy of K. Indeed, if $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$, then $(\text{ad } x)(\text{ad } y)z = [x, [y, z]] \in \mathfrak{a}$ for all $z \in \mathfrak{g}$ (and 0 for all $z \in \mathfrak{a}$. If follows that in the basis $e_1, \dots, e_k$ of $\mathfrak{a}$, $e_1, \dots, e_k, e_{k+1}, \dots, e_n$ basis of $\mathfrak{g}$, the matrix of $(\text{ad } x)(\text{ad } y)$ is of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. But trace of this matrix is 0, so $K(x, y) = 0$.

Conversely, let $\mathfrak{g}$ be semisimple. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. If $K_{\mathfrak{a}\times\mathfrak{a}}$ is degenerate, so that $\mathfrak{a} \cap \mathfrak{a}^\perp \neq 0$, hence $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$ such that $K(\mathfrak{b}, \mathfrak{b}) = 0$. By considering the adjoing representation of $\mathfrak{b}$ in $\mathfrak{g}$ and applying the Cartan criterion we conclude that $\mathfrak{b}$ is solvable. Since $\mathfrak{g}$ is semisimple, we deduce that $\mathfrak{b} = 0$. Thus if $\mathfrak{g}$ is semisimple, the Killing form is non-degenerate, by taking $\mathfrak{a} = \mathfrak{g}$.

As for the second part, we already proved that $K|_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate. Hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. By lemma it is a direct sum of ideals, so $[\mathfrak{a}, \mathfrak{a}^\perp] = 0$. Hence K for $\mathfrak{a}$ equals K of $\mathfrak{g}$ restricted to $\mathfrak{a}$. $\square$

Definition 11.4. *A Lie algebra $\mathfrak{g}$ is called simple if its only ideals are 0 and $\mathfrak{g}$ and $\mathfrak{g}$ is not abelian.*

Corollary 11.1. *Any semisimple, finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0 is a direct sum of simple Lie algebras.*

Proof. If $\mathfrak{g}$ is semisimple, but not simple, and if $\mathfrak{a}$ is an ideal, then by the theorem, the Killing form restricted to $\mathfrak{a}$ is non-degenerate, hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, where $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are also semisimple. After finitely many steps it can be decomposed into simple algebras. $\square$