18.745 Introduction to Lie Algebras	October 12, 2010

Lecture 10 — Trace Form & Cartan's criterion

Prof. Victor Kac — *Scribe: Vinoth Nandakumar*

Definition 10.1. Let $\mathfrak{g}$ be a Lie algebra and π a representation of $\mathfrak{g}$ on a finite dimensional vector space V. The associated trace form is a bilinear form on $\mathfrak{g}$, given by the following formula:

$$(a,b)_V = \text{tr } (\pi(a)\pi(b))$$

Proposition 10.1. *(i) The trace form is symmetric, i.e. $(a,b)_V = (b,a)_V$.*

(ii) The trace form is invariant, i.e. $([a,b],c)_V = (a,[b,c])_V$.

Proof. (i) follows from the fact that tr $(AB) =$ tr (BA). For (ii), note the following:

$$\begin{aligned}([a,b],c)_V &= \text{tr } (\pi([a,b])\pi(c)) = \text{tr } (\pi(a)\pi(b)\pi(c) - \pi(b)\pi(a)\pi(c)) \\ &= \text{tr } (\pi(a)\pi(b)\pi(c) - \pi(a)\pi(c)\pi(b)) = \text{tr } (\pi(a)\pi([b,c])) \\ &= (a,[b,c])_V\end{aligned}$$

□

Definition 10.2. If $\dim \mathfrak{g} < \infty$, then the trace form of the adjoint representation is called the Killing form:

$$\kappa(a,b) = \text{tr } ((\mathbf{ad}\ a)(\mathbf{ad}\ b))$$

Exercise 10.1. Let $\mathbb{F}$ be a field of characteristic 0. Suppose $(\cdot,\cdot)$ be an invariant bilinear form on $\mathfrak{g}$. Show that if $v \in \mathfrak{g}$ is such that **ad** a is nilpotent, then $(e^{\text{ad } v}a, e^{\text{ad } v}b) = (a,b)$. In other words, the bilinear form is invariant with respect to the group G generated by $e^{\text{ad } v}$, **ad** v nilpotent.

Proof.

$$\begin{aligned}(e^{\text{ad } v}a, e^{\text{ad } v}b) &= (a + (\mathbf{ad}\ v)a + \frac{1}{2!}(\mathbf{ad}\ v)^2 + \cdots, b + (\mathbf{ad}\ v)b + \frac{1}{2!}(\mathbf{ad}\ v)^2 b + \cdots) \\ &= (a,b) + \sum_{i\geq 1}\sum_{j+k=i}\frac{1}{j!k!}((\mathbf{ad}\ v)^j a, (\mathbf{ad}\ v)^k b)\end{aligned}$$

Thus it suffices to prove that for $i \geq 1$, $\sum_{j+k=i}\frac{1}{j!k!}((\mathbf{ad}\ v)^j a, (\mathbf{ad}\ v)^k b) = 0$. Note the following:

$$((\mathbf{ad}\ v)^j a, (\mathbf{ad}\ v)^k b) = ((\mathbf{ad}\ v)(\mathbf{ad}\ v)^{j-1}a, (\mathbf{ad}\ v)^k b) = -((\mathbf{ad}\ v)^{j-1}a, (\mathbf{ad}\ v)^{k+1}b)$$

Here we have used the invariance of $(\cdot,\cdot)$. So this means $((\mathbf{ad}\ v)^i a, b) = -((\mathbf{ad}\ v)^{i-1}, (\mathbf{ad}\ v)b) = ((\mathbf{ad}\ v)^{i-2}, (\mathbf{ad}\ v)^2 b) = \cdots$. So:

$$\sum_{j+k=i}\frac{1}{j!k!}((\mathbf{ad}\ v)^j a, (\mathbf{ad}\ v)^k b) = \sum_{j+k=i}\frac{1}{j!k!}(-1)^k((\mathbf{ad}\ v)^i a, b) = \frac{1}{i!}((\mathbf{ad}\ v)^i a, b)\sum(-1)^k\frac{i!}{j!k!} = 0$$

Above we have used the fact that $\sum_{k=0}^{i}(-1)^k\binom{i}{k} = 0$. □

Exercise 10.2. Show that the trace form of $\mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$ associated to the standard representation is non-degenerate and the Killing form on $\mathfrak{sl}_n(\mathbb{F})$ is also non-degenerate, provided char $\mathbb{F} \nmid 2n$. Find the kernel of the Killing form on $\mathfrak{gl}_n(\mathbb{F})$.

Proof. To show the trace form on $\mathfrak{gl}_n(\mathbb{F})$ for the standard representation is non-degenerate, if $\sum_{i,j} a_{ij}e_{ij}$ lies in the kernel where for some r, s, $a_{rs} \neq 0$, then $\mathrm{tr}((\sum_{i,j} a_{ij}e_{ij})e_{sr}) = \mathrm{tr}(a_{rs}e_{rs}e_{sr}) = a_{rs} \neq 0$ since $\mathrm{tr}(e_{i,j}e_{s,r}) = 0$ unless $j = s, i = r$, which is a contradiction. To show the trace form on $\mathfrak{sl}_n(\mathbb{F})$ for the standard representation is non-degenerate, if $x = \sum_{i,j} a_{ij}e_{ij}$ lies in the kernel and some $a_{rs} \neq 0$, then by the same argument we have a contradiction. So say $a_{rs} = 0$ for $r \neq s$, so $x = \sum_{a_{ii}e_{ii}}$. If $a_{jj} \neq a_{kk}$ for some j, k, then $\mathrm{tr}(\sum a_{ii}e_{ii}, e_{jj} - e_{kk}) = a_{jj} - a_k \neq 0$, which is a contradiction. So $a_{jj} = a_{kk} \forall j, k$, so $\mathrm{tr}x = na_{11} = 0$, so $a_{11} = 0$ (since char $\mathbb{F} \nmid n$, and $x = 0$).

For the killing form on $\mathfrak{gl}_n(\mathbb{F})$, consider the basis of $\mathfrak{gl}_n(\mathbb{F})$, $\{e_{ij}\}$. Then:

$$\mathbf{ad}\ e_{ij}\mathbf{ad}\ e_{kl}(e_{gh}) = [e_{ij}, \delta_{lg}e_{kl} - \delta_{lk}e_{gl}] = \delta_{jk}\delta_{lg}e_{kh} - \delta_{ki}\delta_{lg}e_{kj} - \delta_{jg}\delta_{hk}e_{il} + \delta_{li}\delta_{hk}e_{gj}$$

The coefficient of e_{gh} in this expansion is $a_{gh} = \delta_{gi}\delta_{jk}\delta_{lg} - \delta_{jk}\delta_{hg}\delta_{hi}\delta_{lg} - \delta_{gi}\delta_{lh}\delta_{jg}\delta_{hk} + \delta_{hj}\delta_{li}\delta_{hk}$. So:

$$\begin{aligned}\kappa_{\mathfrak{gl}_n}(e_{ij}e_{kl}) &= \sum_{g,h} a_{gh} = \sum_{g,h}(\delta_{gi}\delta_{jk}\delta_{lg} - \delta_{gk}\delta_{hg}\delta_{hi}\delta_{lg} - \delta_{gl}\delta_{kl}\delta_{jg}\delta_{hk} + \delta_{hj}\delta_{li}\delta_{hk}) \\ &= n\delta_{il}\delta_{jk} - \delta_{kl}\delta_{ij} - \delta_{ij}\delta_{kl} + n\delta_{jk}\delta_{il} = 2n\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl} = 2n\mathrm{tr}(e_{ij}e_{kl}) - 2\mathrm{tr}(e_{ij})\mathrm{tr}(e_{kl})\end{aligned}$$

It follows that $\kappa_{\mathfrak{gl}_n}(x, y) = 2n\mathrm{tr}(xy) - 2\mathrm{tr}(x)\mathrm{tr}(y)$ by bilinearity. To calculate the radical of $\kappa_{\mathfrak{gl}_n}$, note if $x = \sum_{i,j} x_{i,j}e_{i,j}$, $\kappa_{\mathfrak{gl}_n}(x, e_{kl}) = 2nx_{lk} - 2(\sum x_{ii})\delta_{kl}$. If this is always 0, $x_{lk} = 0$ when $k \neq l$, and $nx_{kk} = \sum_i x_{ii}$, so $x = \lambda I$ for some λ (since char$\mathbb{F} \nmid 2n$). So the radical of $\kappa_{\mathfrak{gl}_n}$ is $\mathbb{F}I$. By a theorem from Lecture 11, since $\mathfrak{sl}_n(\mathbb{F})$ is an ideal of $\mathfrak{gl}_n(\mathbb{F})$, $\kappa_{\mathfrak{sl}_n}$ is the restriction of $\kappa_{\mathfrak{gl}_n}$ to $\mathfrak{sl}_n(\mathbb{F})$; hence $\kappa_{\mathfrak{sl}_n}(x, y) = 2n\mathrm{tr}(x)\mathrm{tr}(y)$. Since it is a scalar multiple of the trace form of the standard representation, which is non-degenerate, it follows that the radical of $\kappa_{\mathfrak{sl}_n}$ is trivial. □

Lemma 10.2 (Cartan)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{F} = \overline{\mathbb{F}}$, a field of characteristic* 0 *(so that $\mathbb{Q} \subset \mathbb{F}$). Let π be a representation of $\mathfrak{g}$ in a finite dimensional vector space V. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}$, and consider the generalized weight space decomposition of V and the generalized root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$:*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \pi\left(\mathfrak{g}_\alpha\right) V_\lambda \subseteq V_{\lambda+\alpha}, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

Pick $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$, so that $h = [e, f] \in \mathfrak{g}_0 = \mathfrak{h}$. Suppose that $V_\lambda \neq 0$. Then $\lambda(h) = r\alpha(h)$, where $r \in \mathbb{Q}$ depends only on λ and α but not on h.

Proof. Let $U = \bigoplus_{n\in\mathbb{Z}} V_{\lambda+n\alpha} \subset V$. Then dim$U < \infty$, and U is $\pi(e), \pi(f)$ and $\pi(h)$ invariant. But $[\pi(e), \pi(f)] = \pi(h)$, hence $\mathrm{tr}_U(\pi(h)) = 0$. Thus we have the following:

$$0 = \mathrm{tr}_U(\pi(h)) = \sum_n \mathrm{tr}_{V_{\lambda+n\alpha}}(\pi(h)) = \sum_n (\lambda + n\alpha)(h)\mathrm{dim}\ V_{\lambda+n\alpha}$$

In the last line we have used the fact that the matrix of $\pi(h)|_{V_{\lambda+n\alpha}}$ takes the following form:

$$A = \begin{pmatrix} (\lambda+n\alpha)(h) & * & & \\ & (\lambda+n\alpha)(h) & * & \\ & & \ddots & \\ & & & (\lambda+n\alpha)(h) \end{pmatrix}$$

$$\implies \lambda(h)(\sum_n \dim V_{\lambda+n\alpha}) = -\alpha(h)\sum_n n \dim V_{\lambda+n\alpha}$$
$$\implies \lambda(h) = r\alpha(h), r = -\frac{\sum_n n \dim V_{\lambda+n\alpha}}{\sum_n \dim V_{\lambda+n\alpha}}$$

Note in the above that $V_\lambda \neq 0$, so $\sum_n \dim V_{\lambda+n\alpha} \neq 0$. □

Theorem 10.3 (Cartan's criterion)**.** *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, where V is a finite dimensional vector space over $\mathbb{F} = \overline{\mathbb{F}}$, a field of characteristic 0. Then the following are equivalent:*

1. $(\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]) = 0$, *i.e.* $(a,b)_V = 0$ *for* $a \in \mathfrak{g}, b \in [\mathfrak{g},\mathfrak{g}]$.
2. $(a,a)_V = 0$ *for all* $a \in [\mathfrak{g},\mathfrak{g}]$.
3. $\mathfrak{g}$ *is solvable.*

Proof. (i) $\implies$ (ii): Obvious.

(iii) $\implies$ (i): By Lie's theorem, in some basis of V, all matrices of $\mathfrak{g}$ are upper triangular, and thus $[\mathfrak{g},\mathfrak{g}]$ is strictly upper triangular. Thus $\pi(ab)$ is strictly upper triangular and $(a,b)_V = 0$ if $a \in \mathfrak{g}, b \in [\mathfrak{g},\mathfrak{g}]$.

(ii) $\implies$ (iii): Suppose not. Then the derived series of $\mathfrak{g}$ stabilizes, so suppose $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^(k)$ for some k with $\mathfrak{g}^{(k)} \neq 0$. Then $(a,a)_V = 0$ for $a \in \mathfrak{g}^{(k)}$ and $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k)}$. We reach the desired contradiction using the following Lemma. □

Lemma 10.4. *If $\mathfrak{g} \subset \mathfrak{gl}_V$, such that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, then $(a,a)_V \neq 0$ for some $a \in \mathfrak{g}$.*

Proof. Proof by contradiction. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

Since $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, we obtain that $\mathfrak{h} = \sum_{\alpha\in\mathfrak{h}^*}[g_\alpha, g_{-\alpha}]$. Hence by Cartan's Lemma $\lambda(h) = r_{\lambda,\alpha}\alpha(h)$ for $r_{\lambda,\alpha} \in \mathbb{Q}, \neq 0$ if $V_\lambda \neq 0$. The assumption that $(a,a)_V = 0$ means that, for all $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$:

$$0 = (h,h)_V = \mathrm{tr}_V(\pi(h)^2) = \sum_{\lambda\in\mathfrak{h}*} \mathrm{tr}_{V_\lambda}(\pi(h)^2) = \sum_{\lambda\in\mathfrak{h}*} \lambda(h)^2 \dim V_\lambda = \alpha(h)^2 \sum_\lambda r_{\lambda,\alpha}^2 \dim V_\lambda$$

In the above, we have used the fact that $\mathrm{tr}_{V_\lambda}(\pi(h)^2) = \lambda(h)^2 \mathrm{dim}\ V_\lambda$, which follows from the Jordan form theorem, since $\pi(h)|_{V_\lambda}$ can be expressed as an upper triangular matrix with $\lambda(h)$-s on the diagonal. It follows from this calculation that $\alpha(h) = 0$, and hence $\lambda(h) = 0$ for all $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Since $\mathfrak{h} = \sum_{\alpha \in \mathfrak{h}^*}[g_\alpha, g_{-\alpha}]$, it follows that $\lambda(h) = 0$ for all $h \in \mathfrak{h}$. Since $\lambda \in \mathfrak{h}*$ was arbitrary, this means $V = V_0$.

Since $\pi(\mathfrak{g}_\alpha)V = \pi(\mathfrak{g}_\alpha)V_0 \subset V_\alpha = 0$ for $\alpha \neq 0$, it follows that $\mathfrak{g}_\alpha = 0$ for $\alpha \neq 0$. Hence $\mathfrak{g} = \mathfrak{g}_0$, and $\mathfrak{g}$ is nilpotent. This contradicts the fact that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. □

Corollary 10.5. *A finite dimensional Lie algebra* $\mathfrak{g}$ *over* $\mathbb{F} = \overline{\mathbb{F}}$ *is solvable iff* $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Proof. Consider the adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Its kernel is $Z(\mathfrak{g})$. So $\mathfrak{g}$ is solvable iff $\mathrm{ad}\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is a solvable. But by Cartan's criterion, $\mathrm{ad}\mathfrak{g}$ is solvable iff $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$. □

Exercise 10.3. Consider the following 4-dim solvable Lie algebra $D = \mathrm{Heis}_3 + \mathbb{F}d$, where $\mathrm{Heis}_3 = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c$, with the relations $[d, p] = p, [d, q] = -q, [d, c] = 0$. Define on D the bilinear form $(p, q) = (c, d) = 1, \text{rest } = 0$. Show that this is a non-degenerate symmetric invariant bilinear form, but $(D, [D, D]) \neq 0$, so Cartan's criterion fails for this bilinear form.

Proof. It is symmetric by construction. It is nondegenerate since if $a_1p + a_2q + a_3c + a_4d$ lies in the kernel, taking the bilinear form with q, p, d, c respectively gives $a_1 = a_2 = a_3 = a_4 = 0$. Cartan's criterion fails since $(p, q) \neq 0$ and $p \in D, q \in [D, D]$. To check that it is invariant:

$$\begin{aligned}
B([a_1p + a_2q + a_3c + a_4d, &b_1p + b_2q + b_3c + b_4d], c_1p + c_2q + c_3c + c_4d) \\
&= B((a_1b_2 - a_2b_1)c + (a_4b_1 - a_1b_4)p + (a_2b_4 - a_4b_2)q, c_1p + c_2q + c_3c + c_4d) \\
&= -(a_1b_2 - a_2b_1)c_4 + (a_4b_1 - a_1b_4)c_1 + (a_2b_4 - a_4b_2)c_1 \\
B(a_1p + a_2q + a_3c + a_4d, &[b_1p + b_2q + b_3c + b_4d, c_1p + c_2q + c_3c + c_4d]) \\
&= B(a_1p + a_2q + a_3c + a_4d, (b_1c_2 - b_2c_1)c + (b_4c_1 - b_1c_4)p + (b_2c_4 - b_4c_2)q) \\
&= a_1(b_2c_4 - b_4c_2) + a_2(b_4c_1 - b_1c_4) + a_4(b_1c_2 - b_2c_1)
\end{aligned}$$

By comparision, it is clear $B([r, s], t) = B(r, [s, t])$ for $r = a_1p + a_2q + a_3c + a_4d, s = b_1p + b_2q + b_3c + b_4d, t = c_1p + c_2q + c_3c + c_4d$, so B is invariant, as required. □

Remark. Very often one can remove the condition $\mathbb{F} = \overline{\mathbb{F}}$ by the following trick. Let $\mathbb{F} \subset \overline{\mathbb{F}}$ be the algebraic closure. Given a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, let $\overline{\mathfrak{g}} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{g}$ be a Lie algebra over $\overline{\mathbb{F}}$.

Exercise 10.4.

1. $\mathfrak{g}$ is solvable (resp. nilpotent) iff $\overline{\mathfrak{g}}$ is.
2. Derive Cartan's criterion and Corollary for char $\mathbb{F} = 0$ but not $\mathbb{F} = \overline{\mathbb{F}}$.
3. Show that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ is nilpotent iff $\mathfrak{g}$ is solvable when char $\mathbb{F} = 0$
4. $\mathfrak{g}_0^a$ is a Cartan sub-algebra for every regular element of $a \in \mathfrak{g}$ for any field $\mathbb{F}$.

Proof. By construction of $\overline{\mathfrak{g}}$, if we pick a basis $a_1, ..., a_n$ of $\mathfrak{g}$, so that $\mathfrak{g} = \mathbb{F}a_1 + \cdots + \mathbb{F}a_n$, then $\overline{\mathfrak{g}} = \overline{\mathbb{F}}a_1 + \cdots + \overline{\mathbb{F}}a_n$ with the same bracket relations holding.

1. Note $\overline{[\mathfrak{g},\mathfrak{g}]} = [\overline{\mathfrak{g}},\overline{\mathfrak{g}}]$. To see this, $[\mathfrak{g},\mathfrak{g}]$ is the $\mathbb{F}$-span of $[a_i, a_j]$ with certain linear relations holding between them, so both $\overline{[\mathfrak{g},\mathfrak{g}]}$ and $[\overline{\mathfrak{g}},\overline{\mathfrak{g}}]$ are the $\overline{\mathbb{F}}$-span of $[a_i, a_j]$ with certain linear relations holding between them; and the Lie algebra structure is the same. Iterating this, we have $\overline{\mathfrak{g}^{(i)}} = \overline{\mathfrak{g}}^{(i)}$. So $\mathfrak{g}$ is solvable $\leftrightarrow \exists i, \mathfrak{g}^{(i)} = 0 \leftrightarrow \exists i, \overline{\mathfrak{g}}^{(i)} = 0 \leftrightarrow \overline{\mathfrak{g}}$ is solvable. A similar argument shows that $\overline{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]} = [\overline{\mathfrak{g}},[\overline{\mathfrak{g}},\overline{\mathfrak{g}}]]$, and more generally, $\overline{\mathfrak{g}^i} = \overline{g}^i$. So $\mathfrak{g}$ is nilpotent $\leftrightarrow \exists i, \mathfrak{g}^i = 0 \leftrightarrow \exists i, \overline{g}^i = 0 \leftrightarrow \overline{g}$ is nilpotent.

2. In Cartan's Criterion, note that the second condition $(a,a)_V = 0 \forall a \in [\mathfrak{g},\mathfrak{g}]$ is equivalent to the condition $(a,b)_V = 0 \forall a,b \in [\mathfrak{g},\mathfrak{g}]$, since $(\cdot,\cdot)_V$ is symmetric (to see this, expand $(a+b,a+b)_V = 0$ and note it is characteristic 0). Thus by Cartan's criterion, we have $(\mathfrak{g},[\mathfrak{g},\mathfrak{g}])_V = 0 \leftrightarrow (\overline{\mathfrak{g}},[\overline{\mathfrak{g}},\overline{\mathfrak{g}}])_V = 0 \leftrightarrow (a,b)_V = 0 \forall a,b \in [\overline{\mathfrak{g}},\overline{\mathfrak{g}}] \leftrightarrow (a,b)_V = 0 \forall a,b \in [\mathfrak{g},\mathfrak{g}]$. So the first two conditions of Cartan's criterion are equivalent for $\mathrm{char}\mathbb{F} = 0$. By Cartan's criterion, $(a,b)_V = 0 \forall a,b \in [\mathfrak{g},\mathfrak{g}] \leftrightarrow (a,b)_V = 0 \forall a,b \in [\overline{\mathfrak{g}},\overline{\mathfrak{g}}] \leftrightarrow \overline{\mathfrak{g}}$ is solvable $\leftrightarrow \mathfrak{g}$ is solvable, so the last two conditions of Cartan's criterion are equivalent for $\mathrm{char}\mathbb{F} = 0$. For the Corollary, $\mathfrak{g}$ is solvable $\leftrightarrow \overline{\mathfrak{g}}$ is solvable $\leftrightarrow \kappa(\overline{g},[\overline{g},\overline{g}]) = 0 \leftrightarrow \kappa(\mathfrak{g},[\mathfrak{g},\mathfrak{g}]) = 0$, so the Corollary is true for all $\mathrm{char}\mathbb{F} = 0$.

3. This was proven in a previous exercise when $\mathbb{F} = \overline{\mathbb{F}}$. So $\mathfrak{g}$ is solvable $\rightarrow \overline{\mathfrak{g}}$ is solvable $\rightarrow [\overline{\mathfrak{g}},\overline{\mathfrak{g}}]$ is nilpotent $\rightarrow [\mathfrak{g},\mathfrak{g}]$ is nilpotent.

4. a is a regular element of $\mathfrak{g} \implies a$ is a regular element of $\overline{\mathfrak{g}}$ (since the discriminant of a is same in both $\mathfrak{g}$ and $\overline{\mathfrak{g}}$). Hence $\overline{\mathfrak{g}}_0^a$ is a Cartan sub-algebra of $\overline{\mathfrak{g}}$, and hence $\mathfrak{g}_0^a$ is a Cartan sub-algebra of $\mathfrak{g}$. To see the last step, a sub-algebra is Cartan iff it is nilpotent and self-normalizing, and $\overline{\mathfrak{g}_0^a} = \overline{g}_0^a$, so since $\overline{\mathfrak{g}}_0^a$ is nilpotent, $\mathfrak{g}_0^a$ is nilpotent, and $N_{\overline{\mathfrak{g}}}(\overline{g}_0^a) = \overline{\mathfrak{g}}_0^a \rightarrow N_{\mathfrak{g}}(\mathfrak{g}_0^a) = \mathfrak{g}_0^a$.

$\square$