Definition 10.1. Let \( \mathfrak{g} \) be a Lie algebra and \( \pi \) a representation of \( \mathfrak{g} \) on a finite dimensional vector space \( V \). The associated trace form is a bilinear form on \( \mathfrak{g} \), given by the following formula:

\[(a, b)_V = \text{tr}(\pi(a)\pi(b))\]

Proposition 10.1. (i) The trace form is symmetric, i.e. \( (a, b)_V = (b, a)_V \).

(ii) The trace form is invariant, i.e. \( ([a, b], c)_V = ([a, b], c)_V \).

Proof. (i) follows from the fact that \( \text{tr}(AB) = \text{tr}(BA) \). For (ii), note the following:

\[(a, b)_V = \text{tr}(\pi([a, b])\pi(c)) = \text{tr}(\pi(a)\pi(b)\pi(c) - \pi(b)\pi(a)\pi(c)) = \text{tr}(\pi(a)\pi(b)(\pi(c) - \pi(c)\pi(b)) = \text{tr}(\pi(a)\pi([b, c])))\]

\[= (a, [b, c])_V \]

\[\square\]

Definition 10.2. If \( \dim \mathfrak{g} < \infty \), then the trace form of the adjoint representation is called the Killing form:

\[\kappa(a, b) = \text{tr}((\text{ad} \ a)(\text{ad} \ b))\]

Exercise 10.1. Let \( \mathbb{F} \) be a field of characteristic 0. Suppose \( (\cdot, \cdot) \) be an invariant bilinear form on \( \mathfrak{g} \). Show that if \( v \in \mathfrak{g} \) is such that \( \text{ad} \ a \) is nilpotent, then \( (e^{\text{ad} v} a, e^{\text{ad} v} b) = (a, b) \). In other words, the bilinear form is invariant with respect to the group \( G \) generated by \( e^{\text{ad} v} \), \( \text{ad} \ v \) nilpotent.

Proof.

\[(e^{\text{ad} v} a, e^{\text{ad} v} b) = (a + (\text{ad} \ v)a + \frac{1}{2!}(\text{ad} \ v)^2 + \cdots, b + (\text{ad} \ v)b + \frac{1}{2!}(\text{ad} \ v)^2 b + \cdots)\]

\[= (a, b) + \sum_{i \geq 1} \sum_{j+k=i} \frac{1}{j!k!}((\text{ad} \ v)^ja, (\text{ad} \ v)^kb)\]

Thus it suffices to prove that for \( i \geq 1, \sum_{j+k=i} \frac{1}{j!k!}((\text{ad} \ v)^ja, (\text{ad} \ v)^kb) = 0 \). Note the following:

\[((\text{ad} \ v)^ja, (\text{ad} \ v)^kb) = ((\text{ad} \ v)(\text{ad} \ v)^{j-1}a, (\text{ad} \ v)^{k+1}b) = -((\text{ad} \ v)^{j-1}a, (\text{ad} \ v)^{k+1}b)\]

Here we have used the invariance of \( (\cdot, \cdot) \). So this means \( ((\text{ad} \ v)^ja, b) = -((\text{ad} \ v)^{j-1}a, (\text{ad} \ v)^{k+1}b) = ((\text{ad} \ v)^{j-2}, (\text{ad} \ v)^2b) = \cdots \). So:

\[\sum_{j+k=i} \frac{1}{j!k!}((\text{ad} \ v)^ja, (\text{ad} \ v)^kb) = \sum_{j+k=i} \frac{1}{j!k!}((-1)^k((\text{ad} \ v)^ja, b) = \frac{1}{i!}((\text{ad} \ v)^ja, b) \sum_{k=0}^i (-1)^k \frac{i!}{j!k!} = 0\]

Above we have used the fact that \( \sum_{k=0}^i (-1)^k \binom{i}{k} = 0 \). \[\square\]
Exercise 10.2. Show that the trace form of $\mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$ associated to the standard representation is non-degenerate and the Killing form on $\mathfrak{sl}_n(\mathbb{F})$ is also non-degenerate, provided $\text{char } \mathbb{F} \nmid 2n$. Find the kernel of the Killing form on $\mathfrak{gl}_n(\mathbb{F})$.

Proof. To show the trace form on $\mathfrak{gl}_n(\mathbb{F})$ for the standard representation is non-degenerate, if $\sum_{i,j} a_{ij} e_{ij}$ lies in the kernel where for some $r, s$, $a_{rs} = 0$, then $\text{tr}(\sum_{i,j} a_{ij} e_{ij}) = \text{tr}(a_{rs} e_{rs} e_{sr}) = a_{rs} \neq 0$ and $\text{tr}(e_{j} e_{s,r}) = 0$ unless $j = s, i = r$, which is a contradiction. To show the trace form on $\mathfrak{sl}_n(\mathbb{F})$ for the standard representation is non-degenerate, if $x = \sum_{i,j} a_{ij} e_{ij}$ lies in the kernel and some $a_{rs} = 0$, then by the same argument we have a contradiction. So say $a_{rs} = 0$ for $r \neq s$, so $x = \sum_{i,j} a_{ij} e_{ij}$. If $a_{jj} \neq a_{kk}$ for some $j, k$, then $\text{tr}(\sum_{i,l} a_{il} e_{ii}, e_{jj} - e_{kk}) = a_{jj} - a_{kk} \neq 0$, which is a contradiction. So say $a_{jj} = a_{kk} \forall j, k$, so $\text{tr} = \text{tr} = 0$ (since char $\mathbb{F} \nmid n$, and $x = 0$).

For the killing form on $\mathfrak{gl}_n(\mathbb{F})$, consider the basis of $\mathfrak{gl}_n(\mathbb{F}), \{e_{ij}\}$. Then:

$$\text{ad } e_{ij} \text{ad } e_{kl}(e_{gh}) = [e_{ij}, \delta_{il} e_{kl} - \delta_{kl} e_{il}] = \delta_{jk} e_{ig} e_{kh} - \delta_{ki} e_{jg} e_{hk} - \delta_{ij} e_{hk} e_{il} + \delta_{il} e_{jh} e_{gk}$$

The coefficient of $e_{gh}$ in this expansion is $a_{gh} = \delta_{gi} \delta_{jk} \delta_{lg} - \delta_{lj} \delta_{hi} \delta_{ig} \delta_{kg} + \delta_{hi} \delta_{ij} \delta_{hl} \delta_{hk}$. So:

$$\kappa_{\mathfrak{gl}_n}(e_{ij} e_{kl}) = \sum_{g,h} a_{gh} = \sum_{g,h} (\delta_{gi} \delta_{jk} \delta_{lg} - \delta_{lj} \delta_{hi} \delta_{ig} \delta_{kg} + \delta_{hi} \delta_{ij} \delta_{hl} \delta_{hk})$$

$$= n \delta_{il} \delta_{jk} - \delta_{hi} \delta_{ij} - n \delta_{il} \delta_{il} = 2n \delta_{il} \delta_{jk} - 2 \delta_{ij} \delta_{il} = 2n \text{tr}(e_{ij} e_{kl}) - 2 \text{tr}(e_{ij}) \text{tr}(e_{kl})$$

It follows that $\kappa_{\mathfrak{gl}_n}(x, y) = 2n \text{tr}(xy) - 2 \text{tr}(x) \text{tr}(y)$ by bilinearity. To calculate the radical of $\kappa_{\mathfrak{sl}_n}$, note if $x = \sum_{i,j} x_{ij} e_{ij}, \kappa_{\mathfrak{sl}_n}(x, e_{kl}) = 2n x_{kl} - 2 \sum_{i} x_{ii} \delta_{kl}$. If this is always 0, $x_{kl} = 0$ when $k \neq l$, and $n x_{kk} = \sum_{i} x_{ii}$, so $x = \lambda I$ for some $\lambda$ (since char $\mathbb{F} \nmid 2n$). So the radical of $\kappa_{\mathfrak{sl}_n}$ is $\mathbb{F} I$. By a theorem from Lecture 11, since $\mathfrak{sl}_n(\mathbb{F})$ is an ideal of $\mathfrak{gl}_n(\mathbb{F})$, $\kappa_{\mathfrak{sl}_n}$ is the restriction of $\kappa_{\mathfrak{gl}_n}$ to $\mathfrak{sl}_n(\mathbb{F})$; hence $\kappa_{\mathfrak{sl}_n}(x, y) = 2n \text{tr}(x) \text{tr}(y)$. Since it is a scalar multiple of the trace form of the standard representation, which is non-degenerate, it follows that the radical of $\kappa_{\mathfrak{sl}_n}$ is trivial.

Lemma 10.2 (Cartan). Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{F} = \overline{\mathbb{F}}$, a field of characteristic 0 (so that $\mathbb{Q} \subset \mathbb{F}$). Let $\pi$ be a representation of $\mathfrak{g}$ in a finite dimensional vector space $V$. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}$, and consider the generalized weight space decomposition of $V$ and the generalized root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda}, \mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_{\alpha}, \pi(\mathfrak{g}_\alpha) V_{\lambda} \subseteq V_{\lambda+\alpha}, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

Pick $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$, so that $h = [e, f] \in \mathfrak{g}_0 = \mathfrak{h}$. Suppose that $V_{\lambda} \neq 0$. Then $\lambda(h) = r \alpha(h)$, where $r \in \mathbb{Q}$ depends only on $\lambda$ and $\alpha$ but not on $h$.

Proof. Let $U = \bigoplus_{n \in \mathbb{Z}} V_{\lambda+n\alpha} \subset V$. Then $\dim U < \infty$, and $U$ is $\pi(e), \pi(f)$ and $\pi(h)$ invariant. But $[\pi(e), \pi(f)] = \pi(h)$, hence $\text{tr}_U(\pi(h)) = 0$. Thus we have the following:

$$0 = \text{tr}_U(\pi(h)) = \sum_n \text{tr}_{V_{\lambda+n\alpha}}(\pi(h)) = \sum_n (\lambda(n+\alpha)(h) \dim V_{\lambda+n\alpha})$$
In the last line we have used the fact that the matrix of \( \pi(h)|_{V_{\lambda+n\alpha}} \) takes the following form:

\[
A = \begin{pmatrix}
(\lambda + n\alpha)(h) & * & * & \\
(\lambda + n\alpha)(h) & * & \\
& & \ddots & \\
& & & (\lambda + n\alpha)(h)
\end{pmatrix}
\]

\[\implies \lambda(h)(\sum_n \dim V_{\lambda+n\alpha}) = -\alpha(h)\sum_n n \dim V_{\lambda+n\alpha}\]

\[\implies \lambda(h) = r\alpha(h), \quad r = \frac{-\sum_n n \dim V_{\lambda+n\alpha}}{\sum_n \dim V_{\lambda+n\alpha}}\]

Note in the above that \( V_{\lambda} \neq 0 \), so \( \sum_n \dim V_{\lambda+n\alpha} \neq 0 \).

**Theorem 10.3** (Cartan’s criterion). Let \( \mathfrak{g} \) be a subalgebra of \( \mathfrak{gl}(V) \), where \( V \) is a finite dimensional vector space over \( \mathbb{F} = \mathbb{F} \), a field of characteristic 0. Then the following are equivalent:

1. \((\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0\), i.e. \((a, b)_V = 0\) for \( a, b \in [\mathfrak{g}, \mathfrak{g}] \).
2. \((a, a)_V = 0\) for all \( a \in [\mathfrak{g}, \mathfrak{g}] \).
3. \( \mathfrak{g} \) is solvable.

**Proof.** (i) \(\implies\) (ii): Obvious.

(iii) \(\implies\) (i): By Lie’s theorem, in some basis of \( V \), all matrices of \( \mathfrak{g} \) are upper triangular, and thus \([\mathfrak{g}, \mathfrak{g}]\) is strictly upper triangular. Thus \( \pi(ab) \) is strictly upper triangular and \((a, b)_V = 0\) if \( a \in \mathfrak{g}, b \in [\mathfrak{g}, \mathfrak{g}] \).

(ii) \(\implies\) (iii): Suppose not. Then the derived series of \( \mathfrak{g} \) stabilizes, so suppose \([\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k)}\) for some \( k \) with \( \mathfrak{g}^{(k)} \neq 0 \). Then \((a, a)_V = 0\) for \( a \in \mathfrak{g}^{(k)} \) and \([\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k)}\). We reach the desired contradiction using the following Lemma.

**Lemma 10.4.** If \( \mathfrak{g} \subset \mathfrak{gl}_V \), such that \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \), then \((a, a)_V \neq 0\) for some \( a \in \mathfrak{g} \).

**Proof.** Proof by contradiction. Choose a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \), and let:

\[
V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda}, \quad \mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha
\]

Since \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \), and \([\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \), we obtain that \( \mathfrak{h} = \sum_{\alpha \in \mathfrak{h}^*} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \). Hence by Cartan’s Lemma \( \lambda(h) = r_{\lambda,\alpha} \alpha(h) \) for \( r_{\lambda,\alpha} \in \mathbb{Q}, \neq 0 \) if \( V_{\lambda} \neq 0 \). The assumption that \((a, a)_V = 0\) means that, for all \( h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \):

\[
0 = (h, h)_V = \text{tr}_V(\pi(h)^2) = \sum_{\lambda \in \mathfrak{h}^*} \text{tr}_{V_{\lambda}}(\pi(h)^2) = \sum_{\lambda \in \mathfrak{h}^*} \lambda(h)^2 \dim V_{\lambda} = \alpha(h)^2 \sum_{\lambda \in \mathfrak{h}^*} r_{\lambda,\alpha}^2 \dim V_{\lambda}
\]
In the above, we have used the fact that \(\text{tr}_V(\pi(h)^2) = \lambda(h)^2 \dim V\), which follows from the Jordan form theorem, since \(\pi(h)|_V\) can be expressed as an upper triangular matrix with \(\lambda(h)\)-s on the diagonal. It follows from this calculation that \(\alpha(h) = 0\), and hence \(\lambda(h) = 0\) for all \(h \in [g_\alpha, g_{-\alpha}]\).

Since \(h = \sum_{\alpha \in \mathfrak{h}^*} [g_\alpha, g_{-\alpha}]\), it follows that \(\lambda(h) = 0\) for all \(h \in \mathfrak{h}\). Since \(\lambda \in \mathfrak{h}^*\) was arbitrary, this means \(V = V_0\).

Since \(\pi(g_\alpha)V = \pi(g_\alpha)V_0 \subset V_0 = 0\) for \(\alpha \neq 0\), it follows that \(g_\alpha = 0\) for \(\alpha \neq 0\). Hence \(g = g_0\), and \(g\) is nilpotent. This contradicts the fact that \([g, g] = 0\).

**Corollary 10.5.** A finite dimensional Lie algebra \(g\) over \(F = \overline{F}\) is solvable iff \(\kappa(g, [g, g]) = 0\).

**Proof.** Consider the adjoint representation \(g \to \mathfrak{gl}(g)\). Its kernel is \(Z(g)\). So \(g\) is solvable iff \(\text{ad}g \subset \mathfrak{gl}(g)\) is a solvable. But by Cartan’s criterion, \(\text{ad}g\) is solvable iff \(\kappa(g, [g, g]) = 0\).

**Exercise 10.3.** Consider the following 4-dim solvable Lie algebra \(D = \text{Heis}_3 + Fd\), where \(\text{Heis}_3 = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c\), with the relations \([d, p] = p, [d, q] = -q, [d, c] = 0\). Define on \(D\) the bilinear form \((p, q) = (c, d) = 1\), rest \(= 0\). Show that this is a non-degenerate symmetric invariant bilinear form, but \((D, [D, D])\) \(\neq 0\), so Cartan’s criterion fails for this bilinear form.

**Proof.** It is symmetric by construction. It is nondegenerate since if \(a_1p + a_2q + a_3c + a_4d\) lies in the kernel, taking the bilinear form with \(q, p, d, c\) respectively gives \(a_1 = a_2 = a_3 = a_4 = 0\). Cartan’s criterion fails since \((p, q) \neq 0\) and \(p \in D, q \in [D, D]\). To check that it is invariant:

\[
B([a_1p + a_2q + a_3c + a_4d, b_1p + b_2q + b_3c + b_4d], c_1p + c_2q + c_3c + c_4d) \\
= B((a_1b_2 - a_2b_1)c + (a_4b_1 - a_1b_4)p + (a_2b_4 - a_4b_2)q, c_1p + c_2q + c_3c + c_4d) \\
= -(a_1b_2 - a_2b_1)c_1 + (a_4b_1 - a_1b_4)c_1 + (a_2b_4 - a_4b_2)c_1 \\
B(a_1p + a_2q + a_3c + a_4d, [b_1p + b_2q + b_3c + b_4d, c_1p + c_2q + c_3c + c_4d]) \\
= B(a_1p + a_2q + a_3c + a_4d, (b_1c_2 - b_2c_1)c + (b_4c_1 - b_1c_4)p + (b_2c_4 - b_4c_2)q) \\
= a_1(b_2c_4 - b_4c_2) + a_2(b_4c_1 - b_1c_4) + a_4(b_1c_2 - b_2c_1)
\]

By comparison, it is clear \(B([r, s], t) = B(r, [s, t])\) for \(r = a_1p + a_2q + a_3c + a_4d, s = b_1p + b_2q + b_3c + b_4d, t = c_1p + c_2q + c_3c + c_4d\), so \(B\) is invariant as required.

**Remark.** Very often one can remove the condition \(F = \overline{F}\) by the following trick. Let \(F \subset \overline{F}\) be the algebraic closure. Given a Lie algebra \(g\) over \(F\), let \(\overline{g} = \overline{F} \otimes_F g\) be a Lie algebra over \(\overline{F}\).

**Exercise 10.4.**

1. \(g\) is solvable (resp. nilpotent) iff \(\overline{g}\) is.

2. Derive Cartan’s criterion and Corollary for char \(F = 0\) but not \(F = \overline{F}\).

3. Show that \([g, g] = g\) is nilpotent iff \(g\) is solvable when char \(F = 0\).

4. \(g_0^a\) is a Cartan sub-algebra for every regular element of \(a \in g\) for any field \(F\).

**Proof.** By construction of \(\overline{g}\), if we pick a basis \(a_1, ..., a_n\) of \(g\), so that \(g = Fa_1 + \cdots + Fa_n\), then \(\overline{g} = \overline{F}a_1 + \cdots + \overline{F}a_n\) with the same bracket relations holding.
1. Note \([\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}]\). To see this, \([\mathfrak{g}, \mathfrak{g}]\) is the \(\mathbb{F}\)-span of \([a_i, a_j]\) with certain linear relations holding between them, so both \([\mathfrak{g}, \mathfrak{g}]\) and \([\mathfrak{g}, \mathfrak{g}]\) are the \(\mathbb{F}\)-span of \([a_i, a_j]\) with certain linear relations holding between them; and the Lie algebra structure is the same. Iterating this, we have \(\mathfrak{g}^{(i)} = \mathfrak{g}^{(i)}\). So \(\mathfrak{g}\) is solvable \(\iff\exists i, \mathfrak{g}^{(i)} = 0\iff\exists i, \mathfrak{g}^{(i)} = 0\iff\mathfrak{g}\) is solvable. A similar argument shows that \([\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]\), and more generally, \(\mathfrak{g}^i = \mathfrak{g}^i\). So \(\mathfrak{g}\) is nilpotent \(\iff\exists i, \mathfrak{g}^i = 0\iff\exists i, \mathfrak{g}^i = 0\iff\mathfrak{g}\) is nilpotent.

2. In Cartan’s Criterion, note that the second condition \((a,a)_{\mathcal{V}} = 0\forall a \in [\mathfrak{g}, \mathfrak{g}]\) is equivalent to the condition \((a,b)_{\mathcal{V}} = 0\forall a,b \in [\mathfrak{g}, \mathfrak{g}]\), since \((\cdot,\cdot)_{\mathcal{V}}\) is symmetric (to see this, expand \((a+b,a+b)_{\mathcal{V}} = 0\) and note it is characteristic 0). Thus by Cartan’s criterion, we have \((\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_{\mathcal{V}} = 0 \iff ([\mathfrak{g}, \mathfrak{g}], \mathfrak{g})_{\mathcal{V}} = 0 \iff (a,b)_{\mathcal{V}} = 0\forall a,b \in [\mathfrak{g}, \mathfrak{g}] \iff \mathfrak{g}\) is solvable \(\iff\mathfrak{g}\) is solvable, so the last two conditions of Cartan’s criterion are equivalent for \(\text{char } \mathbb{F} = 0\). For the Corollary, \(\mathfrak{g}\) is solvable \(\iff\mathfrak{g}\) is solvable \(\iff \kappa([\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0 \iff \kappa([\mathfrak{g}, \mathfrak{g}]) = 0\), so the Corollary is true for all \(\text{char } \mathbb{F} = 0\).

3. This was proven in a previous exercise when \(\mathbb{F} = \mathbb{F}\). So \(\mathfrak{g}\) is solvable \(\rightarrow\mathfrak{g}\) is solvable \(\rightarrow [\mathfrak{g}, \mathfrak{g}]\) is nilpotent \(\rightarrow [\mathfrak{g}, \mathfrak{g}]\) is nilpotent.

4. \(\mathfrak{a}\) is a regular element of \(\mathfrak{g} \implies \mathfrak{a}\) is a regular element of \(\mathfrak{g}\) (since the discriminant of \(\mathfrak{a}\) is same in both \(\mathfrak{g}\) and \(\mathfrak{g}\)). Hence \(\mathfrak{g}_0^a\) is a Cartan sub-algebra of \(\mathfrak{g}\), and hence \(\mathfrak{g}_0^a\) is a Cartan sub-algebra of \(\mathfrak{g}\). To see the last step, a sub-algebra is Cartan iff it is nilpotent and self-normalizing, and \(\overline{\mathfrak{g}}_0 = \mathfrak{g}_0^a\), so since \(\overline{\mathfrak{g}}_0\) is nilpotent, \(\mathfrak{g}_0^a\) is nilpotent, and \(N_{\text{g}}(\mathfrak{g}_0^a) = N_{\mathfrak{g}}(\mathfrak{g}_0^a) = \mathfrak{g}_0^a\).

\(\square\)