Scott Kovach

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1 Lecture 9

1.1 Chevalley's theorem on Conjugacy of Cartan subalgebras.

Definition 1. *Let A be a nilpotent linear operator on a vector space V over a field $\mathbb{F}$ of char 0. Define $e^A = I + A + A^2/2! + \cdots$. It is finite since A is nilpotent.*

Ex 9.1 If A and B are commuting nilpotent operators, then we know that $A+B$ is a nilpotent operator; show $e^{A+B} = e^A e^B$. In particular, $e^A e^{-A} = I$, hence e^A is invertible.

Ex 9.2 Let $\mathfrak{g}$ be an arbitarary algebra over a field $\mathbb{F}$ of characteristic 0, and let D be a nilpotent derivation of $\mathfrak{g}$. Show that e^D is an automorphism of $\mathfrak{g}$.

Lemma 1. *Assume $\mathbb{F}$ algebraically closed. Let $f : \mathbb{F}^m \to \mathbb{F}^m$ be a polynomial map, that is $f\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_m) \\ \cdots \\ f_m(x_1, \ldots, x_m) \end{pmatrix}$, where the f_i are polynomials. Suppose that for some $a \in \mathbb{F}^m$ the linear map $df|_{x=a} : \mathbb{F}^m \to \mathbb{F}^m$ is non-singular. Then $f(\mathbb{F}^m)$ contains a non-empty Zariski open subset in $\mathbb{F}^m$.*

Proof. Ex 9.3 Show

1. $df|_{x=a}$ is a linear map given by the matrix $(\frac{\partial f_i}{\partial x_j}(a))_{i,j}$.

2. Suppose that $F(f_1(x_1, \ldots, x_m), \ldots, f_m(x_1, \ldots, x_m)) = 0$ for some non-zero polynomial F. Then $\det(\frac{\partial f_i}{\partial x_j}) = 0$ (Hint: apply the chain rule to $\frac{\partial F}{\partial x_j}$).

3. If $y_1, \ldots, y_m \in \mathbb{F}[x_1, \ldots, x_m]$ are algebraicaly independent, then

$$\mathbb{F}(x_1, \ldots, x_m) \supset \mathbb{F}(y_1, \ldots, y_m)$$

   is a finite extension of fields. i.e. each x_i satisfies a non-zero polynomial equation over $\mathbb{F}[y_1, \ldots, y_m]$.

4. Then the Zariski open set we are looking for is the set of points $\{y \in \mathbb{F}^m | p_i(y) \neq 0\}$, where $p_i(y_1, \ldots, y_m)$ is the leading coefficient of the polynomials satisfied by x_i.

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