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1 Lecture 7

1.1 Regular Elements and the rank of a fin-dim lie algebra

Definition 1. *$X = \mathbb{F}^n, \mathbb{F}^n$ a field. The Zariski topology on X is defined by letting closed subsets to be the sets of common zeros of a collection of polynomials $\{P_\alpha\}_{\alpha \in J}$ on $\mathbb{F}^n$.*

A set is closed if and only if it is the set of common zeros of some collection of polynomials.

Ex 7.1: Prove that this is a topological space.

Definition 2. *Fix $a \in \mathfrak{g}$ and consider the characteristic polynomial of an endomorphism ad a, $a \in \mathfrak{g}$, on a finite dimensional Lie algebra $\mathfrak{g}$ of dimension d:*

$$\det_{\mathfrak{g}}(ad\ a - \lambda I_{\mathfrak{g}}) = (-\lambda)^d + c_{d-1}(-\lambda)^{d-1} + \cdots + \det(ada)$$

This is a polynomial of degree d and constant term 0 because $(ad\ a)a = [a, a] = 0$. Hence we can take a and r so that r is minimal such that $c_r \neq 0$:

$$\det_{\mathfrak{g}}(ad\ a - \lambda I_{\mathfrak{g}}) = (-\lambda)^d + c_{d-1}(-\lambda)^{d-1} + \cdots + c_r(-\lambda)^r,$$

where $1 \leq r \leq d$.

The positive integer r is called the rank *of $\mathfrak{g}$. An element $a \in \mathfrak{g}$ is called regular if $c_r(a) \neq 0$. The non-zero polynomial $c_r(a)$ (of degree $d - r$) is called the discriminant of $\mathfrak{g}$.*

Ex 7.2: Show that c_j is a homogeneous polynomial on $\mathfrak{g}$ of degree $d - j$. For example, $c_{d-1} = tr(ad\ a)$

Ex 7.3

1. the Jordan decomposition of ad a is (ad a_s) + (ad a_n) in $gl_n(\mathbb{F})$.

2. If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of a_s then $\lambda_i - \lambda_j$ are eigenvalues of ad a_s.

3. ad a_s has the same eigenvalues as ad a. *Ex 7.4* Deduce that rank $gl_n(\mathbb{F}) = n$, that the discriminant $c_n 9a) = \prod_{i \neq j}(\lambda_i - \lambda_j)$, hence a is regular if all of its eigenvalues are distinct. Compute $c_2(a)$ for $gl_2(\mathbb{F})$ in terms of the matrix coeffs of a.