18.745 Lecture 6 Exercises

September 28, 2010

1

Show that any nonabelian 3-dimensional nilpotent Lie algebra is isomorphic to the Heisenberg algebra H_3.

2

Suppose $\mathbb{F}$ has characteristic 2, and $V = \mathbb{F}[x]/(x^2)$ is a representation of H_3 where $p \mapsto \frac{\partial}{\partial x}$, $q \mapsto x$, and $c \mapsto I$. Then $V = V_\lambda$, but λ is not a linear functional on H_3. Compute λ.

3

By the example of the adjoint representation of a nonabelian solvable Lie algebra, show that the generalized weight space decomposition fails if the Lie algebra is solvable but not nilpotent.

4

Take $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{h} = \{\text{diagonal matrices}\}$. Find the generalized weight space decomposition in both the tautological and the adjoint representations, and check part (b) in the theorem. That is, check the assertion that

$$\pi\left(\mathfrak{g}_\lambda^{\mathfrak{h}}\right) \subseteq V_{\lambda+\alpha}^{\mathfrak{h}}$$
$$\left[\mathfrak{g}_\alpha^{\mathrm{ad}\ \mathfrak{h}}, \mathfrak{g}_\beta^{\mathrm{ad}\ \mathfrak{h}}\right] = \mathfrak{g}_{\alpha+\beta}^{\mathrm{ad}\ \mathfrak{h}}.$$