18.745, Lecture 5, Exercises

Exercise. 5.1 Show that we may relax the assumption on $\mathbb{F}$ in Lie's Lemma. Show Lie's Lemma under the assumptions that $\mathbb{F}$ is algebraically closed and that $\dim V < \mathit{char}\, \mathbb{F}$.

Exercise. 5.2 Consider the Heisenberg algebra H_3 and its representation on $\mathbb{F}[x]$ given by

$$\begin{aligned} c &\mapsto Id, \\ p &\mapsto (f(x) \mapsto xf(x)),\ \forall f(x) \in \mathbb{F}[x], \\ q &\mapsto (f(x) \mapsto \frac{d}{dx} f(x)),\ \forall f(x) \in \mathbb{F}[x]. \end{aligned}$$

Show that the ideal generated by x^N, $0 < N = \mathit{char}\, \mathbb{F}$, in $\mathbb{F}[x]$ is invariant for the representation of H_3 and that the induced representation of H_3 on $\mathbb{F}[x]/(x^N)$ has no weight.

Exercise. 5.3 Show the following two corollaries to Lie's Theorem:

- for all representations π of a solvable Lie algebra $\mathfrak{g}$ on a finite dimensional vector space V over an algebraically closed field $\mathbb{F}$, $\mathit{char}\, \mathbb{F} = 0$, there exists a basis for V for which the matrices of $\pi(\mathfrak{g})$ are upper triangular;
- a solvable subalgebra $\mathfrak{g} \subset \mathfrak{gl}_V$ (V is finite dimensional over an algebraically closed field $\mathbb{F}$, $\mathit{char}\, \mathbb{F} = 0$) is contained in the subalgebra of upper triangular matrices over $\mathbb{F}$ for some basis of V.

Exercise. 5.4 Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over the algebrically closed field $\mathbb{F}$, $\mathit{char}\, \mathbb{F} = 0$. Show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.