18.745 Introduction to Lie Algebras	September 21, 2010

Lecture 4 — Nilpotent and Solvable Lie Algebras

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1.1 Preliminary Definitions and Examples

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The *lower central series* of $\mathfrak{g}$ is the descending chain of subspaces

$$\mathfrak{g}^1 = \mathfrak{g} \supseteq \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] \supseteq \mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2] \supseteq \ldots \supseteq \mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}] \supseteq \ldots$$

while the *derived series* is

$$\mathfrak{g}^{(0)} = \mathfrak{g} \supseteq \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \supseteq \ldots \supseteq \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \supseteq \ldots$$

We note that

(1) $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$ by induction

(2) All $\mathfrak{g}^n$ and $\mathfrak{g}^{(n)}$ are ideals in $\mathfrak{g}$

Definition 1.2. $\mathfrak{g}$ is called *nilpotent* (resp. *solvable*) if $\mathfrak{g}^n = 0$ for some $n > 0$ (resp. $\mathfrak{g}^{(n)} = 0$ for some $n > 0$).

If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}$ is solvable. In fact

$$\{\text{abelian}\} \subsetneq \{\text{nilpotent}\} \subsetneq \{\text{solvable}\}$$

Example 1.1. Let $\mathfrak{g} = \mathbb{F}a + \mathbb{F}b$ with $[a, b] = b$, $\mathfrak{g}^{(1)} = \mathfrak{g}^2 = \mathbb{F}b$, $\mathfrak{g}^3 = \mathfrak{g}^4 = \ldots = \mathbb{F}b$ but $\mathfrak{g}^{(2)} = 0$ so $\mathfrak{g}$ is solvable but not nilpotent.

Example 1.2. Let $\mathbb{H}_3 = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c$ with $[c, \mathfrak{g}] = 0$ and $[p, q] = c$. Then $\mathbb{H}_3^2 = \mathbb{F}c, \mathbb{H}_3^3 = 0$.

Example 1.3.

$$\begin{aligned} gl_n(\mathbb{F}) &\supseteq b_n = \{\text{upper triangular matrices}\} \\ &\supseteq \eta_n = \{\text{strictly upper triangle matrices}\}to \end{aligned}$$

Exercise 1.1. Show b_n is a solvable (but not nilpotent) Lie algebra and that $[b_n, b_n] = \eta_n (n \geq 2)$.

1.2 Simple Facts about Nilpotent and Solvable Lie Algebras

First we note

1. Any subalgebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable).
2. Any factor algebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable)

Exercise 1.2. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be an ideal. Show that if $\mathfrak{h}$ is solvable and $\mathfrak{g}/\mathfrak{h}$ is solvable, then $\mathfrak{g}$ is solvable too.

The last exercise still holds if we everywhere put "nilpotent" in place of "solvable."

Example 1.4. Suppose g= Fa+Fb, $[a, b] = b$. $\mathbb{F}b \subset \mathfrak{g}$ is an ideal, $\mathbb{F}b$ and $\mathfrak{g}/\mathbb{F}b$ are 1-dimensional and hence abelian and nilpotent. But $\mathfrak{g}$ is not nilpotent.

Theorem 1.1. *(a) If $\mathfrak{g}$ is a nonzero nilpotent Lie algebra then $Z(\mathfrak{g})$ is nonzero*

(b) If $\mathfrak{g}$ is a finite-dimensional Lie algebra such that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent.

Proof. (a) Take $N > 0$ minimal such that $\mathfrak{g}^N = 0$. Since $\mathfrak{g} \neq 0$, $N \geq 2$, but then $\mathfrak{g}^{N-1} \neq 0$ and $[\mathfrak{g}, \mathfrak{g}^{N-1}] = \mathfrak{g}^N = 0$, so $\mathfrak{g}^{N-1} \subset Z(\mathfrak{g})$.

(b) $\bar{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, i.e., $\bar{\mathfrak{g}}^n = 0$ for some n which implies $\mathfrak{g}^n \subset Z(\mathfrak{g})$, but then $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subset [\mathfrak{g}, Z(\mathfrak{g})] = 0$.

□

1.3 Engel's Characterization of Nilpotent Lie Algebras

Theorem 1.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent iff for each $a \in \mathfrak{g}$, $(ad\ a)^n = 0$ for some $n > 0$.*

Indeed one may always take $n = \dim\mathfrak{g}$. proof If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}^{n+1} = 0$ for some n. In particular, (ad a)$^n b = 0$ for all $a, b \in \mathfrak{g}$ sinc this is a length $(n+1)$ commutator. For the converse: The adjoint representation gives an injective homomorphism $\mathfrak{g}/Z(\mathfrak{g}) \hookrightarrow gl_{\mathfrak{g}}$ and by assumption the image consists of nilpotent operators. So by Engel's Theorem (from last lecture), $\mathfrak{g}/Z(\mathfrak{g})$ consists of strictly upper triangular matrices in the same basis. Therefore $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent and hence $\mathfrak{g}$ is nilpotent as well.

1.4 How to Classify 2-Step Nilpotent Lie Algebras

Let $\mathfrak{g}$ be n-dimensional and nilpotent with $Z(\mathfrak{g}) \neq 0$ so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent of dimension $n_1 < n$.

Definition 1.3.

- $\mathfrak{g}$ is *1-step nilpotent* if it is abelian
- $\mathfrak{g}$ is *2-step nilpotent* if $\mathfrak{g}/Z(\mathfrak{g})$ is abelian
- $\mathfrak{g}$ is *k-step nilpotent* if $\mathfrak{g}/Z(\mathfrak{g})$ is $(k-1)$-step nilpotent

Let $\mathfrak{g}$ be 2-step nilpotent so $V = \mathfrak{g}/Z(\mathfrak{g})$ is abelian. Consider the bilinear form

$$\begin{gathered} \text{B: V } \times V \to Z(\mathfrak{g}) \\ \text{(a,b)} \mapsto [\tilde{a}, \tilde{b}] \end{gathered}$$

where $\tilde{a}$ and $\tilde{b}$ are preimages of a, b under $\mathfrak{g} \to V$ (B is an *alternating form*, i.e., $B(x, x) = 0$ for all x).

Exercise 1.3. Show that 2-step nilpotent Lie algebras are classified by such alternating bilinear forms.

You can show that the problem of classifying all nilpotent algebras is equivalent to problems that are known to be impossible. However, you can classify things in some special circumstances.

Exercise 1.4. Show that if $Z(\mathfrak{g}) = \mathbb{F}c$ and $\mathfrak{g}$ is 2-step nilpotent, then $\mathfrak{g}$ is isomorphic to $\mathbb{H}_{2n+1} = (\mathbb{F}p_1 + \mathbb{F}p_2 + \ldots + \mathbb{F}p_n) + (\mathbb{F}q_1 + \mathbb{F}q_2 + \ldots + \mathbb{F}q_n) + \mathbb{F}c$ with $[p_i, q_i] = \delta_{ij}$ and $[c, \mathbb{H}_{2k+1}] = 0$.