18.745 Introduction to Lie Algebras	September 14, 2010

Lecture 2 — Some Sources of Lie Algebras

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From Associative Algebras

We saw in the previous lecture that we can form a Lie algebra A_-, from an associative algebra A, with binary operation the commutator bracket $[a, b] = ab - ba$. (The same construction worked for algebras satisfying any one of a variety of other conditions).

As Algebras of Derivations

Lie algebras are often constructed as the subalgebra of derivations of a given algebra. This corresponds to the use of vector fields in geometry.

Definition 2.1. For any algebra A over a field $\mathbb{F}$, a derivation of A is an $\mathbb{F}$-vector space endomorphism D of A satisfying $D(ab) = D(a)b + aD(b)$. Let $\mathrm{Der}(A) \subset \mathfrak{gl}(A)$ be denote the space of derivations of A.

For an element a of a Lie algebra $\mathfrak{g}$, define a map $\mathbf{ad}(a) : \mathfrak{g} \to \mathfrak{g}$, by $b \mapsto [a, b]$. This map is referred to as the adjoint operator. Rewriting the Jacobi identity as

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]], \tag{1}$$

we see that $\mathbf{ad}(a)$ is a derivation of $\mathfrak{g}$. Derivations of this form are referred to as inner derivations of $\mathfrak{g}$.

Proposition 2.1.

(a) $\mathrm{Der}(A)$ *is a subalgebra of* $\mathfrak{gl}(A)$ *(with the usual commutator bracket).*

(b) The inner derivations of a Lie algebra $\mathfrak{g}$ *form an ideal of* $\mathrm{Der}(\mathfrak{g})$. *More precisely,*

$$[D, \mathbf{ad}(a)] = \mathbf{ad}(D(a)) \text{ for all } D \in \mathrm{Der}(a) \text{ and } a \in \mathfrak{g}. \tag{2}$$

Exercise 2.1. Prove (a).

Proof of (b): We check (2) for all derivations D and $a, b \in \mathfrak{g}$:

$$[D, \mathbf{ad}(a)]b = D[a, b] - [a, Db] = [Da, b] = \mathbf{ad}(Da)b,$$

where the second equality holds as D is a derivation. □

From Poisson Brackets

Exercise 2.2. Let $A = \mathbb{F}[x_1, \dots, x_n]$, or let A be the ring of C^∞ functions on $x_1, \dots, x_n$. Define a Poisson bracket on A by:

$$\{f, g\} = \sum_{i,l=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}, \text{ for fixed choices of } \{x_i, x_j\} \in A. \tag{3}$$

Show that this bracket satisfies the axioms of a Lie algebra if and only if $\{x_i, x_j\} = -\{x_j, x_i\}$ and any triple x_i, x_j, x_k satisfy the Jacobi identity.

Example 2.1. Let $A = \mathbb{F}[p_1, \dots, p_n, q_1, \dots, q_n]$. Let $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{p_i, q_j\} = -\{q_i, p_j\} = \delta_{i,j}$. Both conditions clearly hold, and explicitly:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i}.$$

Via Structure Constants

Given a basis $e_1, \dots, e_n$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, the bracket is determined by the structure constants $c_{ij}^k \in \mathbb{F}$, defined by:

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

The structure constants must satisfy the obvious skew-symmetry condition ($c_{ij}^k = -c_{ji}^k$), and a more complicated (quadratic) condition corresponding to the Jacobi identity.

Definition 2.2. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$, be two Lie algebras over $\mathbb{F}$ and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ a linear map. We say that φ is a homomorphism if it preserves the bracket: $\varphi([a, b]) = [\varphi(a), \varphi(b)$, and an isomorphism it is bijective. If φ is an isomorphism, we say that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, written $\mathfrak{g}_1 \cong g_2$.

Exercise 2.3. Let $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be homomorphism. Then:

(a) $\ker \varphi$ is an ideal of $\mathfrak{g}_1$.

(b) im φ is a subalgebra of $\mathfrak{g}_2$.

(c) im $\varphi \cong g_1 / \ker \varphi$.

As the Lie Algebra of an Algebraic (or Lie) Group

Definition 2.3. An algebraic group G over a field $\mathbb{F}$ is a collection $\{P_\alpha\}_{\alpha \in I}$ of polynomials on the space of matrices $\mathrm{Mat}_n(\mathbb{F})$ such that for any unital commutative associative algebra A over $\mathbb{F}$, the set

$$G(A) := \{g \in \mathrm{Mat}_n(A) \mid g \text{ is ivertible, and } P_\alpha(g) = 0 \text{ for all } \alpha \in I\}$$

is a group under matrix multiplication.

Example 2.2. The general linear group GL_n. Let $\{P_\alpha\} = \emptyset$. $\mathrm{GL}_n(A)$ is the set of invertible matrices with entries in A. This is a group for any A, so that GL_n is an algebraic group.

Example 2.3. The special linear group SL_n. Let $\{P_\alpha\} = \{\det(x_{ij}) - 1\}$. $\mathrm{SL}_n(A)$ is the set of invertible matrices with entries in A and determinant 1. This is a group for any A, so that SL_n is an algebraic group.

Exercise 2.4. Given $B \in \mathrm{Mat}_n(\mathbb{F})$, let $O_{n,B}(A) = \{g \in \mathrm{GL}_n(A) : g^{\mathrm{T}} B g = B\}$. Show that this family of groups is given by an algebraic group.

Definition 2.4. Over a given field $\mathbb{F}$, define the algebra of dual numbers D to be

$$D := \mathbb{F}[\epsilon]/(\epsilon^2) = \{a + b\epsilon \mid a, b \in \mathbb{F},\ \epsilon^2 = 0\}.$$

We then define the Lie algebra **Lie** G of an algebraic group G to be

$$\textbf{Lie } G := \{X \in \mathfrak{gl}_n(\mathbb{F}) \mid I_n + \epsilon X \in G(D).$$

Example 2.4. (1) **Lie** $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F})$, sinve $(I_n + \epsilon X)^{-1} = I_n - \epsilon X$. ($I_n - \epsilon X$ approximates the inverse to order two, but over dual numbers, order two is ignored).

(2) **Lie** $\mathrm{SL}_n = \mathfrak{sl}_n(\mathbb{F})$.

(3) **Lie** $O_{n,B} = o_{\mathbb{F}^n,B}$.

Exercise 2.5. Prove (2) and (3) from example 2.4.

Theorem 2.2. ***Lie*** *G is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$.*

Proof. We first show that **Lie** G is a subspace. Indeed, $X \in$ **Lie** G iff $P_\alpha(I_n + \epsilon X) = 0$ for all α. Using the Taylor expansion:

$$P_\alpha(I_n + \epsilon X) = P_\alpha(I_n) + \sum_{i,j} \frac{\partial P_\alpha}{\partial x_{ij}}(I_n)\epsilon x_{ij},$$

as $\epsilon^2 = 0$. Now as $P_\alpha(I_n) = 0$ (every group contains the identity), this condition is linear in the X_{ij}, so that **Lie** G is a subspace.

Now suppose that $X, Y \in$ **Lie** G. We wish to prove that $XY - YX \in$ **Lie** G. We have:

$$I_n + \epsilon X \in G(\mathbb{F}[\epsilon]/(\epsilon^2)), \text{ and } I_n + \epsilon' Y \in G(\mathbb{F}[\epsilon']/((\epsilon')^2)).$$

Viewing these as elements of $G(\mathbb{F}[\epsilon, \epsilon']/(\epsilon^2, (\epsilon')^2))$, we have

$$(I_n + \epsilon X)(I_n + \epsilon' Y)(I_n + \epsilon X)^{-1}(I_n + \epsilon' Y)^{-1} = I_n + \epsilon\epsilon'(XY - YX) \in G(\mathbb{F}[\epsilon, \epsilon']/(\epsilon^2, (\epsilon')^2)).$$

In particular, $I_n + \epsilon\epsilon'(XY - YX) \in G(\mathbb{F}[\epsilon\epsilon']/((\epsilon\epsilon')^2)) = G(D)$, so that $XY - YX \in$ **Lie** G. ☐