Exercises

(1) Verify that the Jacobi identity holds for $[a, b] := ab - ba$ in the following cases:
- 2-member identity: $a(bc) = (ab)c$
- 3-member identity: $a(bc) + b(ca) + c(ab) = (ab)c + (bc)a + (ca)b = 0$
- 4-member identity: $a(bc) - (ab)c = b(ac) - (ba)c$
- 6-member identity: $[a, bc] + [b, ca] + [c, ab] = 0$

(2) Let $sl_n(\mathbb{F}) := \{a \in gl_n(\mathbb{F}) || tr(a) = 0\}$. Check that $sl_n(\mathbb{F})$ is a subalgebra of $gl_n(\mathbb{F})$ i.e. $tr([a, b]) = 0$ for $a, b \in sl_n(\mathbb{F})$.

(3) Let B be a bilinear form on a vector space V. Then $o_{V,B} := \{a \in gl_V | B(a(u), v) = -B(u, a(v))$ for $u, v \in V\}$. Show that $o_{V,B}$ is a subalgebra of the Lie algebra gl_V.

(4) Set $V = \mathbb{F}^n$ and B the matrix of a bilinear form in the standard basis of $\mathbb{F}^n$. Show that $o_{V,B} = \{a \in gl_n(\mathbb{F}) | a^T B + Ba = 0\}$.

(5) Let $f : Mat_n(\mathbb{F}) \to \mathbb{F}$ be a linera function such that $f([a, b]) = 0$ for $a, b \in Mat_n(\mathbb{F})$. For example, the trace function has this property. Show that in general, $f(a) = \lambda tr(a)$ where λ is independent of $a \in Mat_n(\mathbb{F})$.