18.745 Lecture 17 Exercises

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Exercise 17.1

Let

$$\begin{aligned} V &= \bigoplus_{i=1}^{4} \mathbb{R}\epsilon_i \\ (\epsilon_i, \epsilon_j) &= \delta_{ij} \\ Q_{F_4} &= \left\{ \sum_{i=1}^{4} a_i \epsilon_i \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\} \\ \Delta_{F_4} &= \{\alpha \in Q_{F_4} \mid (\alpha, \alpha) = 1 \text{ or } 2\} \,. \end{aligned}$$

Show that (V, Δ_{F_4}) is an indecomposable root system of rank 4 with 48 roots.

Exercise 17.2

Let

$$\begin{aligned} V_{G_2} &= V_{A_2} \\ Q_{G_2} &= Q_{A_2} \\ \Delta_{G_2} &= \{\alpha \in Q_{A_2} \mid (\alpha, \alpha) = 2 \text{ or } 6\} \end{aligned}$$

Show that this is an indecomposable root system with 12 roots.

Exercise 17.3

Prove that if (V, Δ) is an indecomposable root system and $f : V \to \mathbb{R}$ is a linear map such that $f(\alpha) \neq 0$ for all $\alpha \in \Delta$, then there exists a unique highest root $\theta \in \Delta$.

Exercise 17.4

Show that the extended Cartan matrix $\tilde{A}$ satisfies the same properties from the proposition about the Cartan matrix A, except that $\tilde{A}$ has zero determinant. That is, show that

(a) $a_{ii} = 2$, and all $a_{ij} \in \mathbb{Z}$,

(b) $i \neq j \Rightarrow a_{ij} \leq 0$ and $a_{ij} = 0 \iff a_{ji} = 0$, and

(c) if the root system Δ is indecomposable, all proper principal minors of $\tilde{A}$ are positive.

Exercise 17.5

Compute the Dynkin diagrams of both the usual Cartan matrix and the extended Cartan matrix for C_r $(r \geq 2)$ and D_r $(r \geq 3)$.

Exercise 17.6

For each of the rank-2 root systems of type $A_1 \oplus A_1$, A_2, $B_2 (= C_2)$, and G_2, plot the roots as points of a 2-dimensional Euclidean space.