Exercise 16.1. Explain the root lattices in cases C and D.

1. $C_r = sp_{2r}(\mathbb{F})$, $V_{C_r} = \{\sum_{i=1}^r a_i\epsilon_i | a_i \in \mathbb{R}\}$,
   $\Delta_{C_r} = \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j\} \cup \{\pm 2\epsilon_i\}$,
   $Q_{C_r} = \{\sum_{i=1}^r a_i\epsilon_i | a_i \in \mathbb{Z}, \sum_{i=1}^r a_i \in 2\mathbb{Z}\}$.
2. $D_r = so_{2r}(\mathbb{F})$, $V_{D_r} = \{\sum_{i=1}^r a_i\epsilon_i | a_i \in \mathbb{R}\}$,
   $\Delta_{D_r} = \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j\}$,
   $Q_{D_r} = \{\sum_{i=1}^r a_i\epsilon_i | a_i \in \mathbb{Z}, \sum_{i=1}^r a_i \in 2\mathbb{Z}\}$.

Exercise 16.2. Complete the proof of the theorem, for $(\alpha, \beta) = 0$ and $(\alpha, \beta) = 1$.

Theorem 16.1. *Let Q be an even lattice in an Euclidean space V, and assume the subset $\Delta = \{\alpha \in Q | (\alpha, \alpha) = 2\}$ spans V over $\mathbb{R}$. Then (V, Δ) is a root system.*

Exercise 16.3. Show that

1. $\Delta_{E_8} := \{\alpha \in E_8 | (\alpha, \alpha) = 2\} = \{\pm\epsilon_i \pm \epsilon_j | i \neq j\} \cup \big\{\frac{1}{2}(\pm\epsilon_1 ... \pm \epsilon_8) | \text{even number of minus signs}\big\}$,
2. $|\Delta_{E_8}| = 240$,
3. $\mathbb{R}\Delta_{E_8} = V$.

Exercise 16.4. Consider the following subsystem of the root system of type E_8, (V_{E_8}, Δ_{E_8}). Take $\rho = (\frac{1}{2}, ..., \frac{1}{2})$, and let $\Delta_{E_7} = \{\alpha \in \Delta_{E_8} | (\alpha, \rho) = 0\}$, $Q_{E_7} = \{\alpha \in Q_{E_8} | (\alpha, \rho) = 0\}$, $V_{E_7} = \{v \in V_{E_8} | (v, \rho) = 0\}$. Show that

1. (V_{E_7}, Δ_{E_7}) is a root system of rank 7,
2. $|\Delta_{E_7}| = 126$.

Exercise 16.5. Let $\Delta_{E_6} = \{\alpha \in \Delta_{E_7} | (\alpha, \epsilon_7 + \epsilon_8 = 0\}$, $Q_{E_6} = \{\alpha \in Q_{E_7} | (\alpha, \epsilon_7 + \epsilon_8 = 0\}$, $V_{E_6} = \{v \in V_{E_7} | (v, \epsilon_7 + \epsilon_8 = 0\}$. Show that

1. (V_{E_6}, Δ_{E_6}) is a root system of rank 6,
2. $|\Delta_{E_6}| = 72$.