Exercise 15.1. Show $so_N(\mathbb{F}) = \{a \in gl_N(\mathbb{F}) \mid a + a' = 0\}$ where a' is the transposition of a with respect to the anti-diagonal.

Proposition 15.1. *Assume $N \geq 3$, then $so_N(\mathbb{F})$ is semisimple.*

Proof. We show this by the study of the root space decomposition.

Case 1: $N = 2n + 1$ (odd)

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_n & & & & \\ & & & 0 & & & \\ & & & & -a_n & & \\ & & & & & \ddots & \\ & & & & & & -a_1 \end{pmatrix}.$$

Case 2: $N = 2n$ (even)

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & -a_n & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}.$$

In both cases, $a_i \in \mathbb{F}$, $\dim \mathfrak{h} = n$ and $\epsilon_1, ..., \epsilon_n$ form a basis $\mathfrak{h}^*$. Note that $\epsilon_{N+1-j}|_{\mathfrak{h}} = -\epsilon_j|_{\mathfrak{h}}$ and $\epsilon_{\frac{N+1}{2}}|_{\mathfrak{h}} = 0$ if N is odd.

Next, all eigenvectors for $\mathbf{ad}\ \mathfrak{h}$ are elements $e_{i,j} - e_{N+1-j,N+1-i}$ and the root is $\epsilon_i - \epsilon_j|_{\mathfrak{h}}$.

Hence the set of roots is:

$N = 2n + 1 : \Delta_{so_N(\mathbb{F})} = \{\epsilon_i - \epsilon_j, \epsilon_i, -\epsilon_i, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i, j \in \{1, ..., n\}, i \neq j\}$

$N = 2n : \Delta_{so_N(\mathbb{F})} = \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i, j \in \{1, ..., n\}, i \neq j\}$ □

Exercise 15.2. a) Using the root space decomposition, prove that $so_N(\mathbb{F})$ is semisimple if $N \geq 3$.

b) Show $so_N(\mathbb{F})$ is simple if $N = 3$ or $N \geq 5$ by showing that Δ is indecomposable.

Exercise 15.3. Show $\Delta_{so_4(\mathbb{F})} = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\} \sqcup \{\epsilon_1 + \epsilon_2, -\epsilon_1 - \epsilon_2\}$ is the decomposition into indecomposables. Deduce that $so_4(\mathbb{F})$ is isomorphic to $sl_2(\mathbb{F}) \oplus sl_2(\mathbb{F})$.

Exercise 15.4. Repeat the discussion we've done for $so_N(\mathbb{F})$ in the case $sp_{2n}(\mathbb{F})$. First:

$$sp_{2n}(\mathbb{F}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ all } a, b, c, d \ n \text{ x } n \text{ such that } b = b', c = c', d = -a'$$

Next, $\mathfrak{h}$ is the set of all diagonal matrices in $sp_{2n}(\mathbb{F})$

$$\mathfrak{h} = \{ \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ & & & -a_r & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}, a_i \in \mathbb{F}\}$$

Find all eigenvectors for **ad** $\mathfrak{h}$. Show that the set of roots is

$$\Delta_{sp_{sn}(\mathbb{F})} = \{\epsilon_i - \epsilon_j, 2\epsilon_i, -2\epsilon_i, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i, j \in \{1, ..., n\}, i \neq j\}$$

Show always indecomposable and deduce that $sp_{2n}(\mathbb{F})$ simple for all $n \geq 1$.

Exercise 15.5. Let (V, Δ) be a root system. Then Δ is indecomposable if and only if there does not exist non-trivial decomposition $(V, \Delta) = (V_1, \Delta_1) \oplus (V_2, \Delta_2)$ where $V = V_1 \oplus V_2$, $V_1 \perp V_2$, $\Delta_i \subset V_i$, and $\Delta = \Delta_1 \cup \Delta_2$. (Hint: Use String Condition)

Moreover, the decomposition of $\Delta = \bigsqcup \Delta_i$ into indecomposable sets corresponds to decomposition of the root system in the orthogonal direct sum of indecomposable root systems.