1 Lecture 13

Given $\alpha \in \Delta$, pic a non-zero $E \in \mathfrak{g}_\alpha, F \in \mathfrak{g}_{-\alpha}$ st $K(E,F) = 2/K(\alpha,\alpha)$; let $H = 2d^{-1}(\alpha)/K(\alpha,\alpha)$. Then $[H,E] = 2E, [H,F] = -2F, [E,F] = H$. This subalgebra is isomorphic to $\mathfrak{sl}_2$.

Lemma 1. *(Key lemma for sl_2) Let π be a representation of sl_2 in a vector space V over $\mathbb{F}$ of characteristic 0 and let $v \in V$ be a non-zero vector such that $\pi(E)v = 0, \pi(H)v = \lambda v$, for some $\lambda \in \mathbb{F}$. Then*

1. $\pi(HF^n)v = (\lambda - 2n)\pi(F^n)v$
2. $\pi(EF^n)v = n(\lambda - n + 1)\pi(F^{n-1})v$
3. *If $\dim V < \infty$, then λ is a positive integer, the vectors $\pi(F)^j v$ are independent, and $\pi(F)^{\lambda+1} = 0$.*

Exercise 1. *Prove 2 by inducion on $n \geq 1$.*

Exercise 2. *If instead we used $Fv = 0, Hv = \lambda v$, then*

1. $HE^n v = (\lambda + 2n)E^n v$
2. $FE^n v = -n(\lambda + n - 1)E^{n-1}v$
3. *If $\dim V < \infty$, then $-\lambda$ is a positive integer, $\pi(E)^j$ are independent for $0 \leq j \leq -\lambda$, and $E^{-\lambda+1}v = 0$.*

Exercise 3. *Let $g = sl_n(\mathbb{F})$ By a previous ex, K is non-degenerate, so sl_n is semisimple. Find all possibilities for p, q for its root system Δ.*