Lecture 12

Structure Theory of Semisimple Lie Algebras I

Exercise 12.1: Let $\mathfrak{g}$ denote a Lie algebra over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Show that $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is nondegenerate.

Exercise 12.2: Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. Show that there are two distinct conjugacy classes of Cartan subalgebras given by

$$\mathfrak{h}_1 = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathfrak{h}_2 = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Exercise 12.3: Abstract Jordan decomposition in a Lie algebra $\mathfrak{g}$ is unique when it exists if and only if $Z(\mathfrak{g}) = 0$.

Exercise 12.4:

(a) Show that all derivations of the 2-dimensional nonabelian Lie algebra are inner.

(b) Find Der (Heis_3). Note that not all derivations are inner.

Exercise 12.5: Show that Theorem 1 parts (b) and (c), and Theorem 2 part (c) hold for $\mathrm{char}(\mathbb{F}) = 0$. Recall that

Theorem 1. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$ and $\overline{\mathbb{F}} = \mathbb{F}$.

(b) All derivations of $\mathfrak{g}$ are inner.

(c) Any $a \in \mathfrak{g}$ admits a unique Jordan decomposition.

Theorem 2. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$ and $\overline{\mathbb{F}} = \mathbb{F}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra.

(c) $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$.