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1 Lecture 11

1.1 The radical and semisimple Lie algebras

Exercise 1. *Let* $\mathfrak{g}$ *be a Lie algebra. Then*

1. *if* $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ *are ideals, then* $\mathfrak{a} + \mathfrak{b}$ *and* $\mathfrak{a} \cap \mathfrak{b}$ *are ideals, and if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are solvable then* $\mathfrak{a} + \mathfrak{b}$ *and* $\mathfrak{a} \cap \mathfrak{b}$ *are solvable.*

2. *If* $\mathfrak{a}$ *is an ideal and* $b \subset \mathfrak{g}$ *is a subalgebra, then* $\mathfrak{a} + b$ *is a subalgebra*

Exercise 2. *Let* $\mathfrak{h}$ *and* $\mathfrak{r}$ *be Lie algebras and let* $\gamma : \mathfrak{h} \to \mathrm{Der}(\mathfrak{r})$ *be a Lie algebra homomorphism. Let* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ *be the direct sum of vector spaces and extend the bracket on* $\mathfrak{h}$ *and on* $\mathfrak{r}$ *to the whole of* $\mathfrak{g}$ *by letting*

$$[h, r] = -[r, h] = \gamma(h)(r)$$

for $h \in \mathfrak{h}$ *and* $r \in \mathfrak{r}$*. Show that this provides* $\mathfrak{g}$ *with a Lie algebra structure,* $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}$*, and that any semidirect sum of* $\mathfrak{h}$ *and* $\mathfrak{r}$ *is obtained in this way. Finally, show that* $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r}$ *if and only if* $\gamma = 0$.

Exercise 3. *Let* $\mathfrak{g} \subset gl_n(\mathbb{F})$ *be a subspace consisting of matrices with arbitrary first* m *rows and 0 for the rest of the rows. Find* $R(\mathfrak{g})$ *and a Levi decomposition of* $\mathfrak{g}$.

Exercise 4. *Let* V *be finite-dimensional over a field* $\mathbb{F}$ *which is algebraically closed and characteristic 0. Show that* gl_V *and* sl_V *are irreducible subalgebras of* gl_V*. Deduce that* sl_V *is semisimple.*

Exercise 5. *Let* V *be a finite-dimensional vector space with a symmetric bilinear form* $(,)$*. Let* U *be a subspace such that the restriction* $(,)|_{U\times U}$ *is non-degenerate. Denote* $U^{\perp} = \{v \in V | (v, U) = 0\}$*. Then* $V = U \oplus U^{\perp}$.