Scott Kovach

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1 Lecture 10

Exercise 1. *Suppose (a,b) is an invariant bilinear form on* $\mathfrak{g}$. *Show that if* $v \in \mathfrak{g}$ *is such that ad* v *is nilpotent, then* $(e^{ad\, v}a, e^{ad\, v}b) = (a, b)$. *(Hint: Consider the function* $\varphi(t) = (e^{tad\, v}a, e^{tad\, v}b)$. *Show that it is a polynomial in* t *and that* $(d/dt)^k\varphi(t)|t = 0 = 0$.*)*

Definition 1. *Let* $\mathfrak{g}$ *be a Lie algebra and* π *be a representation of* $\mathfrak{g}$ *in a finite-dimensional vector space* V. *The associated trace form is a bilinear form on* $\mathfrak{g}$ *given by the formula* $(a, b)_V = tr_V(\pi(a)\pi(b))$.

Exercise 2. *Show that the trace form on* $\mathfrak{gl}_n(\mathbb{F})$ *and* $\mathfrak{sl}_n(\mathbb{F})$ *associated to the standard representation is non-degenerate provided* $n \neq 2, \mathrm{char}\mathbb{F} \neq 2$.

Likewise, show that the Killing forms are non-degenerate, provided $\mathrm{char}\mathbb{F}$ *does not divide* n.

Exercise 3. *Consider the following 4-dimensional solvable Lie algebra* $D = H_3 + \mathbb{F}d$ *where* $H_3 = p + q + c$ *and* $[d, p] = p, [d, q] = -q, [d, c] = 0$. *Define on* D *the bilinear form* $(p, q) = 1, (c, d) = 1$, *the rest 0. Show that this is a non-degenerate, symmetric, invariant bilinear form, and* $[D, D] = p + q$, *so Cartan's criterion fails for this form.*

Very often one can remove the assumption that $\mathbb{F}$ is algebraically closed. Let $\overline{\mathbb{F}} \supset \mathbb{F}$ be the algebraic closure of $\mathbb{F}$. Given a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, consider $\overline{\mathfrak{g}} = \overline{\mathbb{F}} \otimes \mathfrak{g}$ over $\overline{\mathbb{F}}$. This means the following: choose a basis e_i of $\mathfrak{g}$ over $\mathbb{F}$, so that $[e_i, e_j] = \sum c_{ij}^k e_k$ and let $\overline{\mathfrak{g}} = \sum_i \overline{\mathbb{F}} e_i$ with the same structure constants. Then $\overline{\mathfrak{g}}$ is again a Lie algebra, but over $\overline{\mathbb{F}}$.

Exercise 4. *Prove the following*

1. $\mathfrak{g}$ *is solvable (resp. nilpotent, abelian) if and only if* $\overline{\mathfrak{g}}$ *is.*
2. *Derive Cartan's criterion and the corollary for char* $\mathbb{F} = 0$, *but not algebraically closed.*
3. *Show that* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent if* $\mathfrak{g}$ *is solvable if char* $\mathbb{F} \neq 0$.
4. $\mathfrak{g}_0^a$ *is a Cartan subalgebra of* $\mathfrak{g}$ *for every regular element* $a \in \mathfrak{g}$, *for any* $\mathbb{F}$.