18.745 Problem Set 9

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1 Lecture 19

1. Using the theorem that semisimple Lie algebras are isomorphic iff their Cartan matrices are equal, deduce that

$$\mathfrak{sl}_2 \simeq \mathfrak{so}_3 \simeq \mathfrak{sp}_2, \quad \mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \quad \mathfrak{so}_5 \simeq \mathfrak{sp}_4, \quad \mathfrak{so}_6 \simeq \mathfrak{sl}_4.$$

2. For $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n$, choosing $\mathfrak{h}$ to be the diagonal matrices in $\mathfrak{g}$, for a choice of function f, show that $\mathfrak{n}_+ = \oplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \oplus_{\alpha\in\Delta_-}\mathfrak{g}_\alpha$ are the strictly upper triangular and lower triangular matrices in $\mathfrak{g}$, respectively.

3. Let e_i, f_i, h_i for $1 \le i \le r$ satisfy the relations

   (a) $[h_i, h_j] = 0$

   (b) $[h_i, e_j] = A_{i,j}e_j, [h_i, f_j] = -A_{i,j}f_j$

   (c) $[e_i, f_j] = \delta_{i,j}h_j$

   The Lie algebra generated by these relations is $\widetilde{\mathfrak{g}}(A)$. Let $\widetilde{\mathfrak{n}}_+$, respectively $\widetilde{\mathfrak{n}}_-$, be the subalgebras generated by the e_i, respectively the f_i. Show that

$$[h_i, \widetilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_-] \subset \widetilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_-, [f_i, \widetilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_-] \subset \widetilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_-.$$

4. $\mathfrak{g}(A)$ is obtained by dividing $\widetilde{\mathfrak{g}}(A)$ by its unique proper maximal ideal. Prove that if $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then in $\mathfrak{g}(A)$, the following relations hold:

   (a) $[[e_1, e_2], e_2] = 0$

   (b) $[[e_2, e_1], e_1] = 0$

   and same for the f_is.

2 Lecture 20

1. For a root lattice Q and $\mathfrak{h} = \mathbb{F} \oplus_{\mathbb{Z}} Q$, with roots Δ, we can construct a Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathbb{F}E_\alpha)$, subject to the following rules:

   (a) $[\mathfrak{h}, \mathfrak{h}] = 0$

   (b) $[h, E_\alpha] = (\alpha, h)E_\alpha$ for $h \in \mathfrak{h}, \alpha \in \Delta$

   (c) $[E_\alpha, E_{-\alpha}] = -\alpha$ for $\alpha \in \Delta$

   (d) $[E_\alpha, E_\beta] = 0$ if $\alpha, \beta \in \Delta, \alpha + \beta \notin \Delta \cup \{0\}$

   (e) $[E_\alpha, E_\beta] = \epsilon(\alpha, \beta)E_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in \Delta, \epsilon(\alpha, \beta) \in \mathbb{F}\backslash\{0\}$

If $\epsilon : Q \times Q \to \pm 1$ satisfies

(a) $\epsilon(\alpha, \beta+\gamma) = \epsilon(\alpha,\beta)\epsilon(\alpha,\gamma), \epsilon(\alpha+\beta,\gamma) = \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma)$

(b) $\epsilon(\alpha,\alpha) = (-1)^{(\alpha,\alpha)/2}$

then show that the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ is satisfied if one of x, y, z is in $\mathfrak{h}$.

2. Check the Jacobi identity in the remaining cases, keeping in mind that the case $\alpha+\beta+\gamma \notin \Delta \cup \{0\}$ trivially holds.

3. For E_6 and D_4, we have automorphisms σ of order 2 and 3, respectively, of the Dynkin diagrams and therefore of the indices of e_i, f_i, h_i. Prove that these maps define automorphisms of $\widetilde{\mathfrak{g}}(A)$, and therefore of $\mathfrak{g}(A)$.

4. For (σ_2, E_6) the 4 distinct elements $\frac{1}{2}(X_i + X_{\sigma(i)})$ where $X = E, F, H$ satisfy all Chevalley relations for F_4 and lie in the fixed point set of σ.
Also, for (σ_3, D_4), the 2 distinct elements $\frac{1}{3}(X_i + X_{\sigma(i)} + X_{\sigma^2(i)})$ for $X = E, F, H$ satisfy all Chevalley relations for G_2 and lie in the fixed point set of σ_3.