18.745 Problem Set 8

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1 Lecture 17

1. (a) Let $Q_{E_7} = \{\alpha \in Q_{E_8} | (\alpha, p) = 0\}$ where $p = \frac{1}{2}(\epsilon_1 + \epsilon_2 + ... + \epsilon_8)$, and let $\Delta_{E_7} = \Delta_{E_8} \cup Q_{E_7}$. Describe Δ_{E_7}, and in particular show that it has size 126.

   (b) Let $Q_{E_6} = \{\alpha \in Q_{E_7} | (\alpha, \epsilon_7 + \epsilon_8) = 0\}, \Delta_{E_6} = \Delta_{E_7} \cup Q_{E_6}$. Describe this root system, and show that it has size 72.

   (c) Show that both of the above root systems are indecomposable.

2. Let Q be an integral lattice, and $\Delta = \{\alpha \in Q | (\alpha, \alpha) \in \{1, 2\}\}$. Assume that $\mathbb{R}\Delta = V$. Then (V, Δ) is a root system.

   Bonus Problem: More generally, let Δ be a subset of Q such that it consists of all vectors of given lengths. Assume that all $A_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$. Then (V, Δ) is a root system.

3. Let $\Delta_{F_4} = \{\alpha \in \Gamma_8 | (\alpha, \alpha) \in \{1, 2\}\}$. Note that Δ_{F_4} has size 48 and is the set of $\{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j, \frac{1}{2}(\pm\epsilon_1 \pm ... \pm \epsilon_4\}$. Show that Δ_{F_4} is indecomposable.

4. $V_{G_2} = V_{A_2}$, $Q_{G_2} = Q_{A_2} = \{a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 | a_1 + a_2 + a_3 = 0\}$.

   Let $\Delta_{G_2} = \{\alpha \in Q_{A_2} | (\alpha, \alpha) \in \{2, 6\}\}$. Then (V_{G_2}, Δ_{G_2}) is an indecomposable root system of rank 2. Describe it, and in particular, show that it has 12 roots.

2 Lecture 18

1. Let $\widetilde{A} = \left(\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right)_{i,j=0}^{r}$, where α_0 is the negative of the highest root of, and $\alpha_1, ..., \alpha_r$ are the simple roots of, a root system (V, Δ). Show that $\widetilde{A}$ satisfies the properties of

   (a) $A_{i,i} = 2$

   (b) if $i \neq j$, then $A_{i,j} \leq 0$, and $A_{i,j} = 0$ iff $A_{j,i} = 0$

   (c) All principal minors of A are positive, except $\det A = 0$.

2. For a (possibly extended) Cartan matrix A, 2x2 principal submatrix for indices i and j, we have nodes i and j. If $A_{i,j} = A_{j,i} = -1$, then connect i and j. If $A_{i,j} = -k$, $A_{j,i} = -1$, $k \neq 1$, then i and j with k lines, and draw a line from j to i. This gives a Dynkin diagram.

   For all cases except E_8 and A_r, find the Dynkin diagram for A and $\widetilde{A}$ (so $B_r, C_r, D_r, E_6, E_7, F_4, G_2$).

3. Let A be an r x r matrix, and let B and C be the principal submatrices formed by removing either the first 1 or 2 (for B and C, respectively) rows and columns.

   (a) If $A_{1,1} = 2, A_{1,2} = -a, A_{2,1} = -b$, all other numbers in the first row and column are 0, then $\det A = 2 \det B - ab \det C$.

(b) If $A_{i,i} = c_i, A_{i,i+1} = -a_i, A_{i+1,i} = -b_i$, with the indices being modulo r, then $\det A - \epsilon E_{1,2} = \det A - \epsilon(b_1 \det C + \prod_{i=2}^{r} a_i)$. In particular, if $b_1 > 0$ and all a_i are positive, then $\det(A - \epsilon E_{1,2}) < \det A$.

4. Complete the classification of Dynkin diagrams. We have narrowed it down to diagrams with at least one double connection and no triple connection, and want to show that the only possible ones are B_r, C_r, and F_4.