18.745 Problem Set 7

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October 2018

1 Lecture 15

1. Let char $\mathbb{F} = 0$. Show that $\mathfrak{g}$ over $\mathbb{F}$ is semisimple iff $\bar{\mathfrak{g}} = \bar{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{g}$ is semisimple, where $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$.

2. Let B be the n x n matrix with 1s on the diagonal going from top right to bottom left. Then $\mathfrak{o}_{N,B} = \mathfrak{so}_N(\mathbb{F}) = \{A | A + A' = 0\}$, where A' is the transpose of A from the other diagonal (going from top right to bottom left).

3. (a) Using the root space decomposition, prove that $\mathfrak{so}_N(\mathbb{F})$ is semisimple if $N \geq 3$.
   (b) Show that $_N(\mathbb{F})$ is simple if $N = 3$ or $N \geq 5$ by showing that Δ is indecomposable.

4. Show that the set of roots of $\mathfrak{so}_4(\mathbb{F})$ is $\{\pm(\epsilon_1 - \epsilon_2)\} \sqcup \{\pm(\epsilon_1 + \epsilon_2)\}$, a decomposition in indecomposables. Deduce that $\mathfrak{so}_4(\mathbb{F}) = \mathfrak{sl}_4(\mathbb{F}) \oplus \mathfrak{sl}_4(\mathbb{F})$, a direct sum of ideals.

5. Let K_n be the n x n matrix with 1s on the diagonal going from top right to bottom left. Let B be the $2n$ x $2n$ matrix that is $\begin{bmatrix} 0 & K_n \\ -K_n & 0 \end{bmatrix}$. Then $\mathfrak{o}_{2n,B} = sp_{2n}$.
   (a) $\mathfrak{sp}_{2n}(\mathbb{F}) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | b = b', c = c', d = -a'\}$ for a, b, c, d being n x n matrices.
   (b) Show that the set of roots is $\Delta = \{\epsilon_i - \epsilon_j, \pm 2\epsilon_i, \pm(\epsilon_i + \epsilon_j) | i, j \in [1, 2, ..., n], i \neq j\}$.
   (c) Prove that $\mathfrak{sp}_{2n}(\mathbb{F})$ is semisimple if $2n \geq 2$, and moreover it is simple.

2 Lecture 16

1. Let (V, Δ) be a root system. Then Δ is indecomposable iff there is no decomposition $(V, \Delta) = (V_1, \Delta_1) \oplus (V_2, \Delta_2)$, meaning that $V = V_1 \oplus V_2$ orthogonal direct sum, $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_i \subset V_i$, and both $V_i \neq 0$.

2. For Lie algebras $\mathfrak{sl}_{r+1}(\mathbb{F}), \mathfrak{so}_{2r+1}(\mathbb{F}), \mathfrak{sp}_{2r}(\mathbb{F}), \mathfrak{so}_{2r}(\mathbb{F})$, we take vectors spaces V over $\mathbb{R}$ such that $\mathfrak{sl}_{r+1}(\mathbb{F})$ is the set of $\sum a_i \epsilon_i$ for i going from 1 to $r+1$ with $\sum a_i = 0$, and the other three Lie algebra vector spaces are $\mathbb{R}^r$, spanned by the ϵ_i.

   Prove that the root system lattices of the four Lie algebras are respectively:
   1. $\sum a_i \epsilon_i$ with $\sum a_i = 0, \epsilon_i \in \mathbb{Z}$;
   2. $\sum a_i \epsilon_i$ with $a_i \in \mathbb{Z}$;
   3. $\sum a_i \epsilon_i$ with $a_i \in \mathbb{Z}$ and $\sum a_i$ in $2\mathbb{Z}$;
   and 4. $\sum a_i \epsilon_i$ with $a_i \in \mathbb{Z}$ and $\sum a_i$ in $2\mathbb{Z}$.

3. An integral lattice Q is such that for all $\alpha, \beta \in Q$, $(\alpha, \beta) \in \mathbb{Z}$. Q is called even if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in \mathbb{Z}$.

   For a positive integer r, using $(\epsilon_i, \epsilon_j) = \delta_{i,j}, V = \sum_{i=1}^r \mathbb{R}\epsilon_i$, let

$$\Gamma_r = \{\sum_{i=1}^r a_i \epsilon_i | \text{ either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + 1/2, \sum a_i \in 2\mathbb{Z}.$$

Show that Γ_r is an even lattice iff r is divisible by 8.

4. Consider the lattice $Q_{E_8} = \Gamma_8$. Let $\Delta_{E_8} = \{\alpha \in \Gamma_8 | (\alpha, \alpha) = 2\} = \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm ... \pm \epsilon_8)$ with an even number of minuses$\}$. Then, $(\mathbb{R}\Gamma_8, \Delta_{E_8})$ is a root system. Show that Δ_{E_8} is indecomposable and has size 240.