18.745 Problem Set 6

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1 Lecture 13

1. We have a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. For $\alpha \in \mathfrak{h}^*$, we define $\nu : \mathfrak{h} \to \mathfrak{h}^*$ by $\nu(h)(h') = \kappa(h, h')$, where κ is the Killing form on $\mathfrak{g}$.

   For $0 \neq \alpha \in \mathfrak{h}^*$, pick $E \in \mathfrak{g}_\alpha, F \in \mathfrak{g}_{-\alpha}, H \in \mathfrak{h}$ so that $\kappa(E, F) = \frac{2}{\kappa(\alpha,\alpha)}$, and $H = \frac{2\nu^{-1}(\alpha)}{\kappa(\alpha,\alpha)}$.

   Then $\mathfrak{a}_\alpha = \mathbb{F}E + \mathbb{F}F + \mathbb{F}H$ is a subalgebra of $\mathfrak{g}$ with $[H, E] = E, [H, F] = -2F, [E, F] = H$. For example, $[H, E] = \frac{2}{\kappa(\alpha,\alpha)}[\nu^{-1}(\alpha), E] = \frac{2}{\kappa(\alpha,\alpha)}\alpha(\nu^{-1}(\alpha))E = 2E$.

   Check the relations $[H, F] = -2F, [E, F] = H$ and prove that this Lie algebra is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ by the map $E \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, H \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. Working in the above $\mathfrak{sl}_2(\mathbb{F})$, let π be a representation of the Lie algebra in V, and let v be a vector with $\pi(E)v = 0, \pi(H)v = \lambda v$. Prove that for all positive integers n, $\pi(E)\pi(F)^n v = n(\lambda - n + 1)\pi(F)^{n-1}v$.

3. The opposite of the key $\mathfrak{sl}_2$ lemma. Suppose that $\pi(F)v = 0, \pi(H)v = \lambda v$. Then

   (a) $\pi(H)\pi(E)^n v = (\lambda + 2n)\pi(E)^n v$.

   (b) $\pi(F)\pi(E)^n v = -n(\lambda + n - 1)\pi(E)^{n-1}v, n \in \mathbb{N}$. In particular, if λ is not a nonpositive integer, then all vectors $\pi(E)^n v$ for n nonnegative are linearly independent.

   (c) If the dimension of V is less than ∞, then λ is a nonpositive integer and the vecros $\pi(E)^j v$ for $0 \leq j \leq -\lambda$ are linearly independent, and $\pi(E)^{-\lambda+1}v = 0$.

2 Lecture 14

1. Prove that any symmetric nondegenerate positive semi-definite bilinear form is positive definite using linear algebra.

2. Recall that any semisimple $\mathfrak{g} = \oplus\mathfrak{g}_\mathfrak{i}$ is a direct sum of simple Lie algebras. Show that this decomposition is unique up to permutation, and that any ideal of $\mathfrak{g}$ is a subsum of this sum.

3. Prove that a semisimple Lie algebra being simple implies that its set of nonzero α, Δ, is indecomposable: it cannot be written as a disjoint union of nonempty Δ_1 and Δ_2 such that Δ contains no members of $\Delta_1 + \Delta_2$.

4. Prove that a finite set Δ of nonzero vectors in a vector space is indecomposable iff for any two vectors α and β there exists a sequence of vectors $\gamma_1, ..., \gamma_s$ in Δ, such that $\alpha = \gamma_1, \beta = \gamma_s$, and $\gamma_i + \gamma_{i+1} \in \Delta \cup \{0\}$ for $1 \leq i \leq s - 1$.