18.745 Problem Set 5

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1 Lecture 11

1. (a) The sum of two ideals of a Lie algebra is again an ideal.
   (b) The sum of a subalgebra and an ideal is a subalgebra.

2. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and let $\gamma : \mathfrak{g}_1 \Rightarrow \operatorname{Der}\mathfrak{g}_2$ be a Lie algebra homomorphism. Let $mfg = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of vector spaces, with the same brackets inside $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and $[a, b] = -[b, a] = \gamma(a)(b)$ for $a \in \mathfrak{g}_1, b \in \mathfrak{g}_2$.

   Show that this is a Lie algebra, and any semidirect product of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is obtained in this way. This is a direct product iff $\gamma = 0$.

3. Let $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{F})$ be the subalgebra for which the first m rows are arbitrary, and the rest are 0. Find the Levi decomposition of $\mathfrak{g}$.

4. Let V be a finite-dimensional vector space. Show that $\mathfrak{gl}_V$ and $\mathfrak{sl}_V$ are irreducible. Deduce that $\mathfrak{sl}_V$ is a semisimple Lie algebra if $\operatorname{char}\mathbb{F} = 0$ (in fact if $\operatorname{char}\mathbb{F}$ does not divide $\dim V$).

5. Let V be a finite-dimensional vector space with a nondegenerate symmetric bilinear form $(\, , \,)$, and let U be a subspace of V, such that the restriction of the form to U is nondegenerate. Then $V = U \oplus U^\perp$, where $U^\perp = \{v \in V | (v, U) = 0\}$.

6. A Lie algebra that is the direct sum of simple Lie algebras contains no abelian ideals. A Lie algebra is called simple if it is more than 1 dimensional and it has no ideals different from 0 and itself.

2 Lecture 12

1. Jordan decomposition in a Lie algebra $\mathfrak{g}$ is unique (if it exists) iff $Z(\mathfrak{g}) = 0$.

   Jordan decomposition of an element a is $a = a_s + a_n$ where $\operatorname{ad} a_s$ is diagonalizable, $\operatorname{ad} a_n$ is nilpotent, and $[a_s, a_n] = 0$.

2. (a) Show that all derivations of the 2-dimensional nonabelian Lie algebra are inner.
   (b) Find $\operatorname{Der}\operatorname{Heis}_3$. In particular, show that not all derivations are inner.

3. Show that Theorem 1 (b), (c), and Theorem 2 (c) hold for any field of characteristic 0.
   Theorem 1 (b), (c) state that for $\mathfrak{g}$ a semisimple Lie algebra, all derivations of $\mathfrak{g}$ are inner, and any element a of $\mathfrak{g}$ admits a unique Jordan decomposition.
   Theorem 2 (c) states that for a semisimple $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$, $\mathfrak{h}$ is abelian.